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AN EARLY NINETEENTH CENTURY ARITHMETIC EXERCISE BOOK

By T. M. FLETT

School exercise books have usually a transitory existence. Even if we ourselves retain them as souvenirs of our schooldays—and we seldom do—then our descendants have no such sentimental regard for them, and consign them to the waste-paper basket when we depart this life. A school exercise book of 130 years ago is therefore remarkable for its longevity, as well as interesting for the light it throws on the teachings of that time.

The particular exercise book with which this article is concerned was written by a schoolboy in Wyresdale, about the year 1827. Wyresdale is a little valley some seven miles south-east of Lancaster, on the western slopes of the fells which rise to Bowland Forest. The writer of the book, Edward Winder, was probably brought up at Wyre House, the big house at the lower end of the dale*, and it is possible that he received his education from a tutor engaged by the owners of Wyre House to teach the members of their own family. Thus, although the exercise book illustrates the methods of teaching in use in the early nineteenth century, it may not represent the attainment of a typical country boy at this period.

The exercise book is in the possession of Mr. and Mrs. T. Percy, Caw House, Over-Wyresdale, and I am greatly indebted to them for allowing me to borrow it and to reproduce the extracts given here. They have also very kindly permitted a microfilm of the book to be deposited in the Harold Cohen Library of the University of Liverpool.

The exercise book contains 158 foolscap pages, and is sewn with brown paper wrappings. It covers what would probably now be

* On page 60 of the manuscript there appears a note: "Edward Winder at Wyer House Wyersdale Counting Book".

regarded as a second course in arithmetic, the contents being briefly as follows:

(1) Examples which would now be solved by the unitary method, here solved by the application of various 'rules of three'.

(2) "Practice".

(3) "Numeration and notation" (i.e. the writing of numbers in words).

(4) Addition, subtraction, multiplication, and division of numbers and simple quantities.

(5) Tables of money and various systems of weights and measures.

(6) Addition, subtraction, multiplication, and division of compound quantities.

(7) "Reduction".

(8) Further questions of type (1).

So far as we may judge from the exercise book, the emphasis in Edward Winder's day was on technique, and not on understanding. For each type of problem the teacher enunciated rules, and the pupil had only to learn to apply these, without necessarily understanding why they worked.

The examples (1) on the application of the various 'rules of three' illustrate this outlook. The first four pages of the exercise book contain the solutions of examples on "The Rule of Three Inverse". This rule "Requires the fourth term to be less than the second when the third is greater than the first: or the fourth to be greater than the second when the third is less than the first. Rule. Multiply the first and second together and divide their product by the third: the quotient will be the answer as below".

One of the problems which Edward Winder solved by the application of this rule is of a type which is now classical: "If 108 workmen finish a piece of work in 12 days how many would be sufficient to do it in 3 days". His solution of this problem is as follows:

```
      workmen    days    days
        108   :   12  ::  3
         12
    3 |1296 | 432   men.   Answer
      |12
      ----
          9
          9
          -
          6
          6
          -
```

This problem might have been set at any time during the last two or three hundred years, but others reflect the social conditions which

obtained in 1827. For instance: "If when a peck of wheat is sold for 2s the penny loaf weighs 8oz how much must it weigh when the peck is worth but 1s. 6d."

More complicated problems are dealt with by "The Double Rule of Three". This "is so called because it is composed of 5 numbers given. To find a 6th which if the proportion is direct must Bear such proportion to the 4th and 5th, as the Third bears to the 1st and 2d but if inverse, the 6th numbers must bear such proportion to the 4th and 5th as the first bears to the 2d and 3d the three first terms are supposition, the Two last a demand.

Rule

"Let the principal cause of loss gain interest Or decrease, action or passion be put in the first place. Let that which betokeneth time difference of place and the like be in the second place and the remaining One in the third. Place the other two terms under their like in the supposition. If the Blank falls under the third term multiplyeth first and second terms for a divisor and the other three for a dividend. But If the blank falls under the first or second term multiply the third and fourth terms for a divisor and the Other three for the dividend and the quotient will be the Answer."

Edward Winder's first example on the application of this rule has a familiar ring to it: "If 8 men in 14 days can mow 112 acres of grass how many men can mow 2000 acres in ten days". However, a later example in this same section of the exercise book would be less likely to find its way into present-day school textbooks: "If a family consisting of 7 persons drink 2 kilderkins of beer in 12 days how much will another family of 14 persons drink in 8 days". [A kilderkin contains 18 gallons.] The solution to this latter problem given by Edward Winder is as follows:

persons	days	kilderkins
7	12	2
14	8	

$$\begin{array}{r} 12 \\ 7 \\ \hline 84 \end{array}$$

$$\begin{array}{r} 14 \\ 8 \\ \hline 112 \\ 2 \end{array}$$

$$\begin{array}{r|r|} 84 & 224 \\ & 168 \\ \hline & 56 \end{array} \qquad 2\frac{56}{84} = 12 \text{ gallons Answer.}$$

The next subject dealt with in the exercise book is "Practice... so called from its general use among merchants and tradesmen. It is a concise method of computing the value of articles. &c. by taking aliquot parts. Note. An aliquot part of a number is such a part as being taken a certain number of times will produce the number exactly: thus: 4 is an aliquot part of 12: because 3 fours are 12". Here again the subject is reduced to rules, seven in all. For instance, Rule 1 is: "When the price is less than a penny call the given number pence and take the aliquot parts that are in a penny: then divide by 12 and 20, to reduce the answer to pounds". Later rules seem unnecessarily complicated; for instance, Rule 4 is: "When the price consists of any even number of Shillings under 20 multiply the given quantity by half the price doubling the first figure of the product for shillings and the rest of the product will be pounds". "Practice" was clearly an important part of the curriculum, for Edward Winder covered 12 closely written pages with examples on the use of these rules.

Addition, subtraction, multiplication, and division of numbers and simple quantities (as opposed to compound quantities) could not have been new processes to Edward Winder at this stage, but here too there are neatly written statements of the appropriate rules. For example, under "addition" we find the rule: "Place the numbers one under another, so that units may stand under units tens under tens &c.; add the units, set down the units in their sum and carry the tens as so many ones to the next row; proceed thus to the last row under which set down the whole answer". This process is to be accompanied by a "proof: Begin at the top and add the figures downwards: if the sum is found the same as before it is presumed to be right". Similarly, subtraction is to be checked by addition of "The Difference to the Subtrahend".*

The section on multiplication begins with the standard 12×12 multiplication tables, but proceeds rapidly to such examples as $1701495868567 \times 4767856$. Division proceeds as rapidly. In 14 examples, Edward Winder progressed from such problems as $725107 \div 2$ to $17453798946123741 \div 31479461$.

Many teachers today deplore the time spent in teaching children to use our complicated systems of measures, and would like to see these systems simplified. Few will realize, however, just how far we have travelled along the road to simplification since 1827. Today we deal only with Avoirdupois Weight, but in 1827 Edward Winder had to learn the units in Avoirdupois Weight (drams, ounces,

* Cf. *Ministry of Education Pamphlet* 36, p. 52 "'Checking' should not be regarded as a fad of the teacher, but as essential part of the work. Addition can be checked by adding both up and down; subtraction can be checked by addition; ...".

pounds, stones, quarters, hundreds,† tons), Troy Weight (grains, pennyweights, ounces, pounds) and Apothecaries Weight (grains, scruples, drams, ounces, pounds), as well as those in an unnamed system with 15 lb to the stone, 2 stones to the tod, and 8 tods to the pack or sack.

"Lineal Measure" was almost as complicated. In addition to the inch, foot, yard, pole,‡ land-chain, furlong, and mile, which we still learn about today, Edward Winder had to know the definitions of such units as the barleycorn ($\frac{1}{3}$ in.), fathom (6 ft.), league (3 miles), hand (used in measuring the height of horses, equivalent to 4 in.), pace (5 ft.), and cubit (approximately $1\frac{1}{2}$ ft.). He had also to remember a cloth measure in which the units were the nail ($2\frac{1}{4}$ in.), quarter (9 in.), yard (36 in.), and English, French, and Flemish ells (45, 54, and 27 in., respectively).

In "superficial or square measure" the units known to Edward Winder were the square inch, square foot, square yard, square rod, rood, acre, square link, and square chain, and also the Square. This last, which was equivalent to 100 sq. ft., was used in the measurement of roofing and flooring.

However, complicated as they are, all these systems are simple when compared with the various measures of volume in use in Edward Winder's time. The Imperial Gallon, measuring $277\frac{1}{4}$ cubic inches,§ had just been established by Act of Parliament, thus standardizing the gill, pint, and quart, and also the units of dry measure, namely the peck (2 gallons), bushel (4 pecks), and quarter (8 bushels). In the dry measure which had been used before the introduction of the Imperial Gallon, the gallon contained $268\frac{4}{5}$ cubic inches—and Edward Winder had to learn the conversion factor from the old system to the new. Other units of dry measure mentioned in the exercise book are the coom (? 4 bushels), chaldron (4 quarters), wey (5 quarters) and the last (2 weys). And for coals there was yet another system of units, namely the sack (3 bushels), the chaldron (12 sacks), and the score (21 chaldrons).

Wine and spirit measures introduced further units. The old gallon here contained 231 cubic inches. The units still in use in Edward Winder's time were the hogshead (63 gallons), the pipe or butt (2 hogsheads), and the tun (4 hogsheads). "Some other denominations have been long obsolete: as an anker (10 gallons); a runlet (18 gallons); a tierce (42 gallons); a puncheon (84 gallons);

† That is, hundredweights = 112 lb. But note that cheese and some other goods had formerly been sold in hundredweights of 120 lb.

‡ The exercise book gives also the alternative names "rod" and "perch" for this unit.

§ Actually 277·274 cubic inches. It was defined as the volume of 10lb. avoirdupois of water at 62° F.

But casks of most descriptions are generally charged to the number of gallons contained".

Ale, beer, and porter required yet another system. Here the old gallon contained 282 cubic inches, and other units were the firkin (9 gallons), kilderkin (18 gallons), barrel (36 gallons), hogshead (54 gallons), and butt (108 gallons).

In the exercise book all these systems of measures are used in examples on addition, subtraction, multiplication, and division of compound quantities. For instance, here is an example of addition of various lengths of cloth (Edward Winder's answer is incorrect):

Fl. e	qrs	n
172	2	1
15	1	3
237	0	2
52	1	3
376	2	1
197	1	3
1007	1	1

Again, as an example of multiplication:

beer bar:	fir:	gal:	qts
27	2	4	31
			12
331	2	3	0

Even the "obsolete" units are brought into use to provide further examples. For instance, here is an example of addition of various dry measures:

lasts	wey	q	bu	pks
38	1	4	5	3
47	1	3	6	2
62	0	2	4	3
45	1	4	3	3
78	1	1	2	2
29	1	3	6	2
303	1	0	5	3

The multiplicity of units in use in 1827 allowed Edward Winder's teacher great scope in setting examples on "reduction". Apart from

problems of the form "reduce 17280 farthings into pounds", which could have been taken from twentieth century books, there are such questions as: "In 24 pieces each containing 32 Flemish ells, how many English ells"; "In 5896 gallons of canary how many pipes and hogsheads, of each article an equal number"; and "In 20 chaldrons of coals how many pecks".

In his course of arithmetic Edward Winder learnt how to write invoices and bills, and some of the examples here are of interest in showing the relative prices of different objects.* Here are two examples, both concerned with dress.

London, Sept. 1. 1827

Mr John Thomas

Bought of Samuel Green

	s.	d.	£.	s.	d.
8 pairs of worsted stockings at	4	6	1	16	
5 pairs of thread ditto at	3	2		15	10
3 pairs of black silk ditto at	14	0	2	2	
6 pairs of milled hose at	4	2	1	5	
4 pairs of cotton ditto at	7	6	1	10	
2 yards of fine flannel at	1	8		3	4
			£7	12	2

Wyersdale. March 1830

Mrs Bright

Sold to Marcia Lamb

	s.	d.	£.	s.	d.
18 yards of French lace at	12	3	11	0	6
5 pairs of fine kid gloves at	2	2		10	10
1 dozen French fans at	3	6	2	2	
2 surperb silk shawls at three guineas each	63		6	6	
4 dozen Irish lamb at	1	3	3	0	
6 sets of knots at	2	6		15	
			£23	14	4

The final section of the exercise book deals with further problems of the type which would now be solved by the unitary method. The appropriate rule here is "The single Rule of three Direct", which "Teacheth by three numbers given to find out a fourth in such proportion to the third as the second is to the first". The rule itself is charmingly, if somewhat obscurely, stated in rhyme.

* The prices may, of course, have been a few years out of date in 1827.

"Three numbers being given, we are taught
By this plain rule to find a fourth that's sought
One of the given numbers does contain
The very question ask'd and should maintain
The third position in the stated ese (?)
The like it first, the others second place
The first and third must then be made the same
The second brought into its lowest name
That done then multiply the latter two
And by the first divide, when, to your view,
The quotient will present the answer true
and its denomination is the same
as that in which you left your second in".

Edward Winder filled the last 19 pages of his exercise book with problems on the application of this rule, a typical one being: "If 27 yards of holland cost £5. 12. 6 how many English ells can I buy for £100".

The last of these problems—the last in the book—is also stated in rhyme. Perhaps the reader may like to apply the rhymed "single Rule of three Direct" to this problem to obtain his own answer.

"As I was beating on the Forest ground
Up starts a hare before my two grayhounds
The Dogs being light of foot did foreby run
Unto her fifteen Roods* just twenty one.
The distance that she started up before
Was four score, sixteen Roods, just and no more
Now this I'd have you unto me declare
How far they ran before they caught the hare".

T. M. F.

University of Liverpool

* Presumably 'rods'.

HENRY BRIGGS: THE BINOMIAL THEOREM ANTICIPATED

By D. T. Whiteside

One of the pleasant aspects of research into mathematical history is the way in which existing material, passed over in the conventional account, may allow us not only to establish the bare, if unexpected, fact of a priority but, more importantly, to assess the significance of major currents in mathematical thought with greater precision. Briggs' partial discovery, in anticipation of Newton, of the general binomial expansion,

$$(1 + x)^\rho = \lim_{n\to\infty} \sum_{0\leq\lambda\leq n}\left[\binom{\rho}{\lambda} \times x^\lambda\right], \quad x < 1,$$

$$= 1 + \rho x + \frac{\rho(\rho-1)}{1 \times 2} x^2 + \frac{\rho(\rho-1)(\rho-2)}{1 \times 2 \times 3} x^3 + \ldots,$$

is an excellent case in point.

Henry Briggs (1561–1631), who held chairs of geometry at Gresham College in London and at Oxford successively, is one of the great figures in early 17th century English mathematics, but remembered today above all for his work in logarithmic computation. The story of his unbounded admiration for the theory of logarithms proposed by the mystic-mathematician John Napier is familiar. His thoughts afire from his first reading of the work in which Napier explained the construction of his logarithmic canon*—"Neper... hath set my head and hands at work with his new and admirable logarithms...I never saw a book which pleased me better and made me more wonder,"—he journeyed to Scotland to discuss logarithms with their inventor. In their conversation was born the common logarithm as we know it today—a concept which improves on Napier's own first construction by having $\log_{10}(1) = 0$ and $\log_{10}(10) = 1$ while retaining the essential property that $\log_{10}(\alpha) + \log_{10}(\beta) = \log_{10}(\alpha \times \beta) + [\log_{10}(1) = 0]$. Napier himself died shortly afterward, but after years of work Briggs finally published his monumental "Arithmetica logarithmica" (London, 1624), which tabulates the common logarithms, to 14 decimal places, of the numbers 1 to 10,000 and 90,000 to 100,000.

From a theoretical viewpoint and particularly now that the tables themselves have been long superseded what is supremely interesting in the "Arithmetica logarithmica" is the lengthy introduction in which Briggs explains his methods of construction. Clearly if

* "Mirifici logarithmorum canonis descriptio", Edinburgh, 1614, an account amplified in the complementary (and posthumous) "constructio" of 1619.

we have suitable interpolation methods only a relatively small number of logarithms need be calculated from first principles, and Briggs' preface with its wide variety of subtabulation methods forms, in effect, the first treatise in the calculus of finite differences. For the calculation of these basic logarithms Briggs could, of course, employ Napier's construction, but that involved the tricky computation of numerous geometrical proportionals and Briggs preferred to use a second method of his own. Observing that, for small x, $\log_e (1 + x) = x$ very nearly, and that any (positive) number L can be made to differ from unity by as little as we please by extracting its square root successively a sufficient number of times ($L^{1/2^r} \approx 1$ for r sufficiently great) Briggs formulates the rule that, for great enough r, $\log_e (L^{1/2^r}) = 1/2^r \times \log_e (L) \approx L^{1/2^r} - 1$. In a numerical example he takes $L = 1.00776$, deriving

$$
\begin{aligned}
L^{1/2} &= 1\cdot 00387\ 72833\ 36962\ 45663 \ldots \\
L^{1/4} &= 1\cdot 00193\ 67661\ 36946\ 61675 \ldots \\
L^{1/8} &= 1\cdot 00096\ 79146\ 39097\ 01728 \ldots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
L^{1/2^8} &= 1\cdot 00003\ 02321\ 60505\ 63775 \ldots,
\end{aligned}
$$

so that

$$\log_e (1\cdot 00776) \approx 2^8 \times (0\cdot 00003\ 02321\ 60505\ 63775).$$

Briggs' method is remarkably accurate, but the successive square root extractions become rapidly tedious, and quite impracticable in large-scale computation. However Briggs, examining the sequences of successive square roots of several particular numbers L, noticed* that the "Briggsian" n-order differences,

$$\delta_s{}^1 = \frac{1}{2} \times (L^{1/2^s} - 1) - (L^{1/2^{s+1}} - 1)$$

$$\delta_s{}^2 = \frac{1}{4} \times \delta_s{}^1 - \delta^1_{s+1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\delta_s{}^n = \frac{1}{2^n} \times \delta_s^{n-1} - \delta_{s+1}^{n-1}$$

form a swiftly decreasing numerical sequence, and so "induced" that this holds generally.

Reformulating Briggs' further argument (which is largely verbal and uses the archaic Bombelli ring notation for variables) let us define the sequence $e_i = (1 + \alpha)^{2^i} - 1$, $i = -1, 0, 1, 2, \ldots$;

* "Arithmetica logarithmica": chapter 8: pp. 17–19.

$$
\begin{aligned}
e_{-1} &= (1+\alpha)^{1/2} - 1,\\
e_0 &= (1+\alpha)^1 - 1 = \alpha,\\
e_1 &= (1+\alpha)^2 - 1 = 2\alpha + \alpha^2,\\
e_2 &= (1+\alpha)^4 - 1 = 4\alpha + 6\alpha^2 + 4\alpha^3 + \alpha^4,\\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

Further, we introduce an n-order "Briggsian" difference on the lines suggested by the numerical examples—specifically,

$$
\begin{cases}
\Delta_i^1 = \dfrac{1}{2} \times e_{i+1} - e_i,\\[2ex]
\Delta_i^\lambda = \dfrac{1}{2^\lambda} \times \Delta_{i+1}^{\lambda-1} - \Delta_i^{\lambda-1}, \quad \lambda = 2, 3, 4, \dots.
\end{cases}
$$

From this recursion Briggs was able to tabulate successively

$$
\begin{aligned}
\Delta_0^1 &= \frac{1}{2}\alpha^2,\\
\Delta_0^2 &= \frac{1}{2}\alpha^3 + \frac{1}{8}\alpha^4,\\
\Delta_0^3 &= \frac{7}{8}\alpha^4 + \frac{7}{8}\alpha^5 + \frac{7}{16}\alpha^6 + \frac{1}{8}\alpha^7 + \frac{1}{64}\alpha^8,\\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

(his table extends up to Δ_0^{10}). By inspection the rule jumps to the eye that Δ_0^λ contains no terms in α^r, $r \leqslant \lambda$, and this Briggs accepted without any further justification. In order, finally, to extract the square root of $(1+\alpha)^{1/2}$ it remains only to set up a sufficient number of the differences Δ_i^λ and so form an appropriate difference table from which $e_{-1} = (1+\alpha)^{1/2} - 1$ may be calculated. In fact, (though this last step is not explicitly taken by Briggs) we can derive a general square root expansion by "unwrapping" the various n-order differences in an obvious way:

$$
\begin{aligned}
\Delta_{-1}^1 &= \frac{1}{2} \times e_0 - e, &\text{or}\quad e_{-1} &= \frac{1}{2} \times e_0 - \Delta_{-1}^1,\\
\Delta_{-1}^2 &= \frac{1}{4} \times \Delta_0^1 - \Delta_{-1}^1 &\text{or}\quad \Delta_{-1}^1 &= \frac{1}{4} \times \Delta_0^1 - \Delta_{-1}^2,\\
\Delta_{-1}^3 &= \frac{1}{8} \times \Delta_0^2 - \Delta_{-1}^2, &\text{or}\quad \Delta_{-1}^2 &= \frac{1}{8} \times \Delta_0^2 - \Delta_{-1}^3,\\
&\dots\dots\dots\dots, & &\dots\dots\dots\dots,
\end{aligned}
$$

and so derive

$$e_{-1} = \frac{1}{2} \times e_0 - \frac{1}{4}\,\Delta_{-1}^1 = \frac{1}{2} \times e_0 - \left(\frac{1}{4} \times \Delta_0^1 - \Delta_{-1}^2\right) = \ldots,$$

$$= \frac{1}{2} \times e_0 - \frac{1}{4} \times \Delta_0^1 + \frac{1}{8} \times \Delta_0^2 - \ldots;$$

or, substituting $e_{-1} = (1+\alpha)^{1/2} - 1$, $e_0 = \alpha$, $\Delta_0^1 = \frac{1}{2}\,\alpha^2, \ldots$, and collecting powers of α,

$$(1+\alpha)^{1/2} - 1 = \frac{1}{2}\,\alpha - \frac{1}{8}\,\alpha^2 + \frac{1}{16}\,\alpha^3 - \frac{5}{128}\,\alpha^4 + \ldots,$$

which is the binomial expansion. [A similar method yields, from the sequence $f_i = (1+\alpha)^{\rho^i} - 1$ and the n-order differences

$$\begin{cases} \Delta_i^1 = \dfrac{1}{\rho} \times f_{i+1} - f_i, \\[2ex] \Delta_i^\lambda = \dfrac{1}{\rho^\lambda} \times \Delta_{i+1}^{\lambda-1} - \Delta_i^{\lambda-1}, \end{cases}$$

the corresponding binomial expansion of $((1+\alpha)^{1/\rho} - 1)$].

It is satisfying to recognise Briggs' genius, especially when the conventional account tends to belittle it. But we should not after all be surprised that Briggs, immersed as he was for many years in the practical interpolation of the logarithmic function, should stumble over a binomial expansion. As his manuscripts of the period make abundantly clear Newton, away in Boothby in Lincolnshire in the Plague Year, 1665, was to come across the general binomial expansion in an analogous way, "interpolating" it equally from a set of known numerical instances. While there is nothing in all the numerous manuscripts which seems to show that Newton ever read Briggs' work (or the similar preface with which Briggs introduces his trigonometrical canon, "Trigonometria britannica"; Gouda, 1633)—and that the highly meticulous annotator which was the young Newton would have passed it by without comment is not to be believed—, shall we not see there a kindred spirit, a like feeling for mathematical structure, reaching over the years?.

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THE DENSITY OF PRIME NUMBERS

By G. Hoffman de Visme

Proofs of the prime number theorem are extremely hard to follow, and leave the impression, at least among amateurs, that the essential property of prime numbers, namely their primeness, plays very little part in the argument.

Simple reasoning, based on the rules of probability, can however give a very fair indication of the way in which the density of primes in the region x varies with x.

If one examines the list of primes, one observes that, although the average density of primes declines steadily, the primes themselves appear to be distributed quite randomly. This is because whether or not a number is prime depends on whether or not it is divisible by one or more of *all* the primes less than its square root, and this latter series of primes forms a distribution which repeats itself within a range vastly greater than the number itself.

Let us therefore ask ourselves the question: "What is the probability that a number x selected at random shall be prime?"

Assuming the divisibility of a number by various primes to be entirely independent "events", e.g. the fact of a number being divisible by 31 is in no wise related to its divisibility by 41, the answer to the question is: 'The product of the probabilities that it shall not be divisible by 2, 3, 5, 7, etc., up to the greatest prime less than its square root', i.e.:—

$$(1-1/2)(1-1/3)(\ldots)(1-1/p),$$

where p is the largest prime less than $x^{1/2}$.

If we compare the average density of primes in region x with that predicted by this argument, we find that the predicted density is rather greater than the actual density in a proportion which at worst is about 1·1. This inequality in a sense expresses the error in the assumption of randomness made above.

Evidently such an error must yield a corresponding error in the value of $\pi(x)$ calculated on this basis since the average density of primes in region x is the average rate of change of $\pi(x)$ in region x, but nevertheless it is worth pursuing the argument further so that we may compare its results with the known asymptotic formula for $\pi(x)$, viz. $(x/\log x)$.

Suppose a smooth curve $y(x)$ which best fits the actual variation of $\pi(x)$ with x. In fact, in the realm of large numbers the proportional deviation of $\pi(x)$ from such a curve is very small indeed.

The slope of this curve at $x = x^{1/2}$ represents the density of primes in the region of $x^{1/2}$, and so the reciprocal of this slope gives

the average interval between primes in that region. Let this interval be h.

By the above argument then, we have

$$\text{Slope in region } (x^{1/2}+h)^2 = (1 - 1/x^{1/2})y'(x)$$

identifying $x^{1/2}$ with the greatest prime less than $(x^{1/2}+h)$, a fair approximation involving negligible error for x large.

Dividing, we get

$$(1 - 1/x^{1/2}) = \text{slope in region } (x^{1/2}+h)^2)/(\text{slope in region } x),$$

that is

$$y'((x^{1/2}+h)^2)/y'(x).$$

For large values of x the numerator can be expressed as

$$y'(x) + (2hx^{1/2})y''(x),$$

and so

$$2x \,.\, y''(x)/y'(x) + y'(x^{1/2}) = 0,$$

remembering that h is the reciprocal of the slope at $x^{1/2}$.

If we substitute the value $(x/\log x)$ for y, we find that the first term comes to

$$\frac{2(2-\log x)}{\log x(\log x - 1)}$$

while the second term comes to

$$\frac{-2(2-\log x)}{(\log x)^2},$$

the sum of these two terms differing from zero only by reason of the difference between $(\log x - 1)^{-1}$ and $(\log x)^{-1}$.

Exact agreement could hardly have been expected since the asymptotic formula gives consistently low values for $\pi(x)$, while this approach, since it yields gradients which are too high, would give too high a value for $\pi(x)$. Nevertheless this approach does serve to explain the shape of the $\pi(x)$ versus x curve, even if it cannot give its exact value. It does also rule out the form $x^{(1-d)}$, with d indefinitely small, for $\pi(x)$.

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A FUNCTIONAL EQUATION IN THE HEURISTIC THEORY OF PRIMES

By E. M. Wright

In the preceding note Mr de Visme writes $y'(x)$ for the "density" of primes in the neighbourhood of x and proves heuristically that

$$2x\{y''(x)/y'(x)\} + y'(x^{1/2}) = 0. \tag{1}$$

Let us put

$$\alpha = \log 2, \log x = 2^v, \quad y'(x) \log x = 1 + w(v), \tag{2}$$

so that, if $w(v) \to 0$ as $v \to \infty$ we have $y'(x) \sim 1/\log x$ as $x \to \infty$. This is equivalent to the prime number theorem. Clearly

$$\frac{dx}{dv} = \alpha x \log x$$

and so

$$\begin{aligned}\frac{w'(v)}{1 + w(v)} &= \alpha x \log x \left\{\frac{y''(x)}{y(x)} + \frac{1}{x \log x}\right\} \\ &= \alpha\{1 + 2^v x y''(x)/y'(x)\} \\ &= \alpha\{1 - 2^{v-1} y'(x^{1/2})\}\end{aligned}$$

by (1). But

$$1 + w(v - 1) = y'(x^{1/2}) \log x^{1/2} = 2^{v-1} y'(x^{1/2})$$

and so

$$w'(v) = -\alpha w(v - 1)\{1 + w(v)\}. \tag{3}$$

I have studied equation (3) for general positive α at some length elsewhere [4] and, in particular, have proved that $w(v) \to 0$ as $v \to +\infty$ when $0 < \alpha \leq 3/2$. For $0 < \alpha < 1$ the proof is fairly simple and this covers the case $\alpha = \log 2$.

This relationship between the equation satisfied by de Visme's $y'(x)$ and (3), which I have studied in such detail, is not a coincidence. About 1942 Lord Cherwell, by reasoning very similar to de Visme's, obtained (1). He showed it to me and asked if I could deduce that $y'(x) \sim 1/\log x$. My attempts to do so led to equation (3), which seemed to have some interest for general positive α. Equation (3) has also been investigated independently by other authors [2, 3], who encountered it in a different application.

Lord Cherwell never published this part of his work but, by different reasoning, he found another, rather more intractable, functional equation for the density of primes [1]. He would have been very interested in Mr. de Visme's note and would, in particular, have sympathised very much with the first sentence, which roughly

represents his own views and his motive for trying to find a more transparent proof of the Prime Number Theorem.

In [4] I ascribe my first interest in (3) to Lord Cherwell's finding an equivalent equation in his attempts to prove the Prime Number Theorem. Several people have asked me just how he came across this equation in the theory of primes and I could not answer them, since I had long ago forgotten his argument and destroyed his letter. But Mr. de Visme's re-discovery has now made this clear.

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[3] S. Kakutani and L. Markus, On the non-linear difference differential equation $y'(t) = [A - By(t-1)]y(t)$, *Theory of Nonlinear Oscillations IV* (*Annals of Math. Studies* No. 41, Princeton 1958), 1–18.

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GLEANINGS FAR AND NEAR

1953. "He (Robert Boyle) deliberately tried to discipline his mind . . . and he found that a very effective way of doing so was to turn to problems of algebra which needed his whole concentration, or to extract square and cube roots in his head. As he was never very good at mathematics he no doubt found this serious occupation a good remedy!

Robert Boyle, Father of Chemistry by Roger Pilkington. p. 36. [Per Mr. J. Buchanan.]

1954. "He (Robert Boyle) would steal away from the gay clamour of the Earl's household and roam alone for hour after hour—lost in strange and unrecorded thoughts and deliberately turning to problems of arithmetic to recall his wandering attention back to the present."

Robert Boyle, Father of Chemistry by Roger Pilkington. p. 39. [Per Mr. J. Buchanan.]

1955. A tells the truth 3 times out of 4, B 4 out of 5, and C 6 out of 7. What is the probability of an event which A and B assert has taken place and which C denies?—A's capacity for truth is as 3 to 1 (lie), B as 4 to 1, C as 6 to 1. If A and B make a joint assertion the odds are as 7 to 2 in its favour. The odds in favour of C's denial are 6 to 1 (or 12 to 2). With the common factor 2 the chances are 12 to 7 in favour of C's denial.—*Daily Mail*, 21st September, 1959. [Per Mr. I. C. Bridges.]

CHANGE OF VARIABLE IN RIEMANN INTEGRATION

By H. Kestelman

1. *Introduction.* If F and G are differentiable functions of one variable, the "function of a function" rule is

$$\frac{d}{dx} F\{G(x)\} = f\{G(x)\}g(x), \tag{1}$$

where f and g denote the derivatives of F and G respectively. When integration is defined as the inverse of differentiation, this justifies the formula

$$F\{G(b)\} - F\{G(a)\} = \int_a^b f\{G(t)\}g(t)\, dt, \tag{2}$$

and we have the substitution rule

$$\int_{G(a)}^{G(b)} f(x)\, dx = \int_a^b f\{G(t)\}g(t)\, dt, \tag{3}$$

valid provided $G'(t) = g(t)$ for $a \leq t \leq b$ and f has a primitive F (that is, $F' = f$) in $G([a, b])$. (If G is any function defined on a set S, we shall denote by $G(S)$ the set of numbers $G(x)$ with x in S.)

Instead of supposing that f and g have primitives, assume now that

(i) *g is Riemann-integrable over* $[a, b]$ (see section 2),

(ii) *G is an indefinite integral of g*, that is

$$G(t) = \int_h^t g(w)\, dw \qquad (a \leq t \leq b), \tag{4}$$

h being some fixed number in $[a, b]$, and

(iii) *f is Riemann-integrable over* $G([a, b])$.

Then (3) is the formula for change of variable in Riemann integration. However, although these hypotheses on f and g imply the existence of the left-hand side of (3), it is not obvious that they imply even the existence of the integral on the right hand side. In a recent review (*Math. Gaz.* Vol. XLIV, No. 347, p. 73) the question was raised as to what additional restrictions must be placed on f and g, to justify (3). The main object of the present note is to show that no additional restrictions are needed:

Theorem. *Under the hypotheses* (i), (ii) *and* (iii), *the function* $f\{G(t)\}g(t)$ *is Riemann-integrable over* $[a, b]$ *and* (3) *holds.*

The proof occupies section 4.

If f and g are everywhere continuous, then their indefinite integrals F and G are at the same time primitive functions, and the previous

proof of (3) applies. (Here G is defined by (4), and for every x in $G([a, b])$

$$F(x) = \int_k^x f(u)\,du, \tag{5}$$

where k is any fixed number in $G([a, b])$.) The proof is harder, but still elementary, if f alone is assumed continuous or if g is assumed to have constant sign (see section 3). But in the general case the difficulty of the proof is similar to that for the Lebesgue integral; indeed, once the integrability of $f\{G(t)\}g(t)$ is known, (3) may be deduced from the result for Lebesgue integrals. However, in this note no use will be made of Lebesgue integration.

It is interesting to note that although $f\{G(t)\}g(t)$ turns out to be necessarily integrable over $[a, b]$, the function $f\{G(t)\}$ does not (see section 5); this is what makes the general proof difficult.

2. *Integration.* Before proceeding to the proofs, we recall some basic properties of Riemann integration, in which the integral is defined as the limit of a sum. The notation introduced in this section will be used throughout.

If $a < b$, the "open interval" (a, b) means the set of all numbers t with $a < t < b$, and the "closed interval" $[a, b]$ that of all t with $a \leq t \leq b$; if Δ denotes one of these then its length $b - a$ is denoted by $|\Delta|$.

A function ψ which is bounded on $[a, b]$ is said to be integrable over $[a, b]$ if there exists a number L with the following property: given any $\varepsilon > 0$ there exists $\delta > 0$ such that for every chain of numbers

$$a = x_0 \leq \xi_1 \leq x_1 \leq \xi_2 \leq x_2 \leq \dots \leq x_{n-1} \leq \xi_n \leq x_n = b \tag{6}$$

with

$$0 < x_r - x_{r-1} < \delta \qquad (1 \leq r \leq n)$$

we have

$$\left| L - \sum_{r=1}^{n} \psi(\xi_r)|\Delta_r| \right| < \varepsilon, \tag{7}$$

where $\Delta_r = [x_{r-1}, x_r]$. If ψ is integrable then L is denoted by $\int_a^b \psi(x)\,dx$.

Denoting by $\omega(\psi;\ \Delta_r)$ the excess of the upper over the lower bound of ψ in Δ_r, it follows that if ψ is integrable over $[a, b]$ then

$$\sum_{r=1}^{n} \omega(\psi;\ \Delta_r)|\Delta_r| \leq 2\varepsilon \qquad \text{if } |\Delta_r| < \delta \quad (1 \leq r \leq n), \tag{8}$$

and that if $a \leq q < p \leq b$ then for all x in $[q, p]$,

$$\left|\frac{1}{p-q}\int_q^p \psi(t)\,dt - \psi(x)\right| \leq \omega(\psi;\ [q,p]). \tag{9}$$

It is easily deduced that if k is any fixed number in $[a, b]$ and

$$\Psi(x) = \int_k^x \psi(t)\,dt,$$

then

$$\Psi'(X) = \psi(X) \tag{10}$$

for every number X in (a, b) at which ψ is continuous, and

$$|\Psi(p) - \Psi(q)| \leq |p - q| M_\psi, \tag{11}$$

where M_ψ is the upper bound of $|\psi(x)|$ for $a \leq x \leq b$.

If $a = b$ then we set $\int_a^b \psi(t)\,dt = 0$, while if $b < a$ we set

$$\int_a^b \psi(t)\,dt = -\int_b^a \psi(t)\,dt.$$

The conditions for ψ to be integrable may be most concisely expressed in terms of "null sets". A set S of real numbers is called "null" if given any $\varepsilon > 0$ there exists a sequence of intervals $I_1, I_2, \ldots$ covering S and such that $|I_1| + |I_2| + \ldots < \varepsilon$. Note that the union of a sequence of null sets is itself null. A statement about a variable number x is said to hold *almost everywhere* in some set E if the numbers x in E for which the statement is false form a null set. With this definition, we can state the following theorem due to Lebesgue: *ψ is integrable over $[a, b]$ if and only if ψ is bounded in $[a, b]$ and is continuous almost everywhere in $[a, b]$.* (For proof, see for example H. Kestelman: *Modern Theories of Integration*)

3. *Special cases.* We assume from now on that conditions (i), (ii) and (iii) of section 1 are satisfied, and we set $\phi(x) = f\{G(x)\}$.

If ϕ is integrable over $[a, b]$, we can give a quite simple proof of (3). We must prove that for the chain (6), the difference

$$\left|\int_{G(a)}^{G(b)} f(x)\,dx - \sum_{r=1}^{n} \phi(\xi_r) g(\xi_r) |\Delta_r|\right| \tag{12}$$

is arbitrarily small if $\Delta_1, \ldots, \Delta_n$ are all small enough.

Now for each r from 1 to n,

$$\int_{G(x_{r-1})}^{G(x_r)} f(x)\,dx - \phi(\xi_r) g(\xi_r) |\Delta_r| = \int_{G(x_{r-1})}^{G(x_r)} [f(x) - f\{G(\xi_r)\}]\,dx \tag{13}$$
$$+ \phi(\xi_r)\int_{x_{r-1}}^{x_r} [g(x) - g(\xi_r)]\,dx.$$

Since every number from $G(x_{r-1})$ to $G(x_r)$ is in $G(\Delta_r)$, the magnitude of the first integral on the right hand side of (13) cannot exceed

$$\omega(\phi;\ \Delta_r)|G(x_r) - G(x_{r-1})| \leq \omega(\phi;\ \Delta_r)M_g|\Delta_r| \quad \text{(by (11))}.$$

The magnitude of the second integral cannot exceed $M_\phi\omega(g;\ \Delta_r)|\Delta_r|$. Hence (12) cannot exceed

$$M_g \sum_{r=1}^{n} \omega(\phi;\ \Delta_r)|\Delta_r| + M_\phi \sum_{r=1}^{n} \omega(g;\ \Delta_r)|\Delta_r|.$$

Since ϕ and g are integrable over $[a, b]$, the result now follows from the analogues of (8). In particular, if f is continuous on $G([a, b])$ then ϕ, being a continuous function of a continuous function, is continuous and therefore integrable, and hence (3) holds.

If f has discontinuities but $g(x) \geq 0$, then

$$G(a) \leq G(\xi_1) \leq G(x_1) \leq \ldots \leq G(x_{r-1}) \leq G(\xi_r)$$
$$\leq G(x_r) \leq \ldots \leq G(b)$$

and so

$$\left| \sum_{r=1}^{n} \int_{G(x_{r-1})}^{G(x_r)} [f(x) - f\{G(\xi_r)\}]\, dx \right|$$
$$\leq \sum_{r=1}^{n} \{G(x_r) - G(x_{r-1})\}\omega(f;\ [G(x_{r-1}), G(x_r)]).$$

Applying this to (13), and using the analogues of (8) corresponding to the integrability of f over $G([a, b])$ and g over $[a, b]$, we deduce that (12) is arbitrarily small if $\Delta_1, \ldots, \Delta_n$ are all small enough.

We shall now prove (3) without assuming that ϕ is integrable.

4. *Proof of the main theorem.* We shall need the four Lemmas which follow.

Lemma 1. *If $\Phi(x)$ is defined for $a \leq x \leq b$, and if*

$$|\Phi(p) - \Phi(q)| \leq K|p - q|$$

whenever $a \leq p < q \leq b$, where K is a constant, then $\Phi(S)$ is null for every null set S contained in $[a, b]$.

Proof. Given $\varepsilon > 0$ there exists a sequence of intervals $J_1, J_2, \ldots$ covering S and such that $\sum_{r=1}^{\infty} |J_r| < \varepsilon$. Then $\Phi(J_1), \Phi(J_2), \ldots$ are intervals (some of which may degenerate into single points) covering $\Phi(S)$ and

$$\sum_{r=1}^{\infty} |\Phi(J_r)| \leq \sum_{r=1}^{\infty} K|J_r| < \varepsilon K.$$

Consequently $\Phi(S)$ is a null set.

LEMMA 2. *With G as defined by* (4), *if S is a subset of* $[a, b]$ *for which $G(S)$ is null, then $g(x) = 0$ almost everywhere in S.*

Proof. Let S_+ be the set of those X in S such that g is continuous at X and $g(X) > 0$. If X is in S_+ then X is interior to a closed interval J with rational end points in which g has a *positive* lower bound, say m. This implies that G is strictly increasing in J and that G^{-1}, the inverse of G, satisfies the conditions of Lemma 1 in $G(J)$, with $K = 1/m$. Hence, since $G(S_+)$ is null, the part of S_+ in J is null, and thus S_+ is the union of a sequence of null sets.

A similar argument shows that S_-, the set of numbers in S at which g is continuous and negative, is a null set. Finally, the numbers X in S which are not in S_+ or S_- and for which $g(X) \neq 0$ are points at which g is discontinuous, and these also form a null set. Hence $g(x) = 0$ almost everywhere in S.

LEMMA 3. *With F and G as defined by* (5) *and* (4),

$$\frac{d}{dx} F\{G(x)\} = f\{G(x)\}g(x) \tag{14}$$

and

$$f\{G(x)\}g(x) \quad \textit{is continuous almost everywhere in } [a, b]. \tag{15}$$

Proof. Since g is continuous almost everywhere in $[a, b]$, it is enough to show that (14) and (15) hold for almost all x at which g is continuous. Now if g is continuous at x then (14) and (15) are obviously true if f is continuous at $G(x)$, and equally if f is discontinuous at $G(x)$ provided $g(x) = 0$, since

$$\begin{aligned} |F\{G(x+h)\} - F\{G(x)\}| &= \left| \int_{G(x)}^{G(x+h)} f(u)\, du \right| \\ &\leq |G(x+h) - G(x)| M_f, \end{aligned} \tag{16}$$

while $G'(x) = g(x) = 0$. We have thus only to show that the set S of those x such that $g(x) \neq 0$ and f is discontinuous at $G(x)$ is a null set. But this follows from Lemma 2 since the integrability of f implies that $G(S)$ is a null set.

LEMMA 4. *If $\Phi(x)$ satisfies the conditions of Lemma* 1, *and if $\Phi'(x) = 0$ almost everywhere in* $[a, b]$, *then $\Phi(b) = \Phi(a)$.*

Proof. Suppose if possible that $\Phi(b) > \Phi(a)$. Choose a positive number ε so small that, with $\Psi(x) = \Phi(x) - \varepsilon(x - a)$, we have $\Psi(b) > \Psi(a)$. Clearly $\Psi(x)$ also satisfies the conditions of Lemma 1, and $\Psi'(x) = -\varepsilon < 0$ except on a null set S. Then by Lemma 1, $\Psi(S)$ is a null set, and we can choose in the interval $(\Psi(a), \Psi(b))$ a number η not in $\Psi(S)$. Since Ψ is continuous, there exists in (a, b) at least one solution of the equation $\Psi(x) = \eta$, and there is a greatest solution, say $x = \xi$. Since $\Psi(\xi) = \eta < \Psi(b)$, it follows

that $\Psi(x) > \Psi(\xi)$ for all x in (ξ, b); on the other hand $\Psi'(\xi) < 0$ since ξ is not in S, and this is a contradiction.

The hypothesis $\Phi(b) < \Phi(a)$ leads similarly to a contradiction; hence $\Phi(b) = \Phi(a)$.

PROOF OF (3). By Lemma 3, $f\{G(x)\}g(x)$ is Riemann-integrable over $[a, b]$. Let

$$\Phi(x) = \int_{G(a)}^{G(x)} f(u)\,du - \int_a^x f\{G(t)\}\,g(t)\,dt.$$

By Lemma 3, $\Phi'(x) = 0$ almost everywhere in $[a, b]$, and from (16) and (11) it follows that $\Phi(x)$ satisfies the conditions of Lemma 1. Consequently, by Lemma 4 we have $\Phi(b) = \Phi(a) = 0$, in other words (3) holds.

5. *A counter-example.* We shall now show by means of an example that although the conditions (i), (ii) and (iii) of section 1 imply the integrability of $f\{G(t)\}g(t)$, they do not imply that of $f\{G(t)\}$, even if $g(t)$ is continuous (or even differentiable).

The rational numbers in $[0, 1]$ can be arranged in a sequence and can therefore be covered by a sequence of open intervals whose lengths have a sum less than 1; these intervals form an open set U consisting of disjoint open intervals (a_1, b_1), (a_2, b_2), ... with $\sum_r (b_r - a_r) < 1$. The points of $[0, 1]$ not in U form a set C which is not null (for if it were null then $[0, 1]$ could be covered by intervals of total length less than one, which is impossible).

We next define a continuous function $g(x)$ such that

$$(17) \qquad G(x) = \int_0^x g(t)\,dt = 0 \textit{ if and only if } x \textit{ is in } C.$$

This may be done by setting $g(x) = \sum_{r=1}^{\infty} g_r(x)$, where

$$(18) \qquad g_r(x) = \frac{1}{2^r} \sin\left\{\left(\frac{x - a_r}{b - a_r}\right) 2\pi\right\} \quad \text{if } a_r \leq x \leq b_r$$

and $g_r(x) = 0$ otherwise. Since the series is uniformly convergent in $[0, 1]$, g is continuous in $[0, 1]$ and

$$G(x) = \sum_{r=1}^{\infty} \int_0^x g_r(t)\,dt \quad \text{if } 0 \leq x \leq 1;$$

but it is easily seen from the graph of g_r that $\int_0^x g_r(t)\,dt$ is positive if $a_r < x < b_r$ and is zero otherwise, and this justifies (17).

Now define $f(x)$ to be 1 if $1/x$ is an integer and zero otherwise. Then f is Riemann-integrable over every interval. To show that $f\{G(x)\}$ is not Riemann-integrable over $[0, 1]$, it is enough to show that this function is discontinuous at every x in C. Now if X is in

C then $G(X) = 0$ and so $f\{G(X)\} = 0$; but if $h > 0$ then $(X, X + h)$ includes rational numbers and therefore numbers ξ in U. By (17), $G(\xi) \neq 0$, and since G is continuous there exists in (X, ξ) a number η such that $G(\eta) = 1/n$ where n is an integer. Since $f\{G(\eta)\} = 1$ this implies that $f\{G(x)\}$ is discontinuous at $x = X$.

It is not hard to show that if $g_r(x)$ defined by (18) is modified by multiplying it by $(x - a_r)^2(x - b_r)^2$ then the corresponding function g is *differentiable* and G still satisfies (17).

The author is much indebted to Dr. Roy Davies for suggestions which have clarified the presentation of this note.

University College, London H. K.

AN ELEMENTARY PROOF OF THE THEOREM ON CHANGE OF VARIABLE IN RIEMANN INTEGRATION

By Roy O. Davies

1. The preceding paper considers the most general theorem on change of variable in a Riemann integral: *If $g(t)$ is integrable over $[a, b]$ and $f(x)$ is integrable over $G([a, b])$, then $\int_a^b f(G(t))g(t)\,dt$ exists and equals $\int_{G(a)}^{G(b)} f(x)\,dx$.*

This can be derived quite easily from the proof of the corresponding result for Lebesgue integrals (the statement of which may be obtained by replacing the word 'integrable' by the word 'summable' and adding the hypothesis that $\int_{G(a)}^{G(t)} f(x)\,dx$ be an absolutely continuous function of t), but the proof of the latter is difficult (see Rogosinski, Volume and Integral, pp. 153–156; Graves, The Theory of Functions of Real Variables, pp. 221–223). The object of Kestelman's paper was to give an independent proof, using from the Lebesgue theory only the concept of a null set (set of measure zero). However, even this concept seems alien to the Riemann theory, so the following elementary proof may be of interest. It is based on essentially the same ideas.

2. *Conventions.* All integration is understood to be in the Riemann sense. If $g(t)$ is integrable over $[a, b]$, then $G(t)$ denotes an indefinite integral of $g(t)$, that is,

$$G(x) = \int_c^x g(t)\,dt \quad (a \leq x \leq b),$$

where c is some fixed number in $[a, b]$. Then $G(x)$ is continuous on $[a, b]$, and the set of numbers $G(x)$ for $a \leq x \leq b$ also forms a closed interval, denoted by $G([a, b])$. (This will have $G(a)$, $G(b)$ as its end-points if $G(x)$ is monotonic, but not necessarily otherwise.) The oscillation (upper bound minus lower bound) of $f(x)$ in an interval is denoted by $osc[f(x)]$. By a *subdivision* of $[a, b]$ we understand a subdivision into a finite number of non-overlapping closed intervals.

We shall use the following standard TEST FOR INTEGRABILITY: *$f(x)$ is integrable over $[a, b]$ if and only if it is bounded and given any $\varepsilon, \eta > 0$ there exists a subdivision of $[a, b]$ such that the intervals in which $osc[f(x)] > \eta$ have total length $< \varepsilon$.*

3. *Proof of the Theorem.* Let

$$M = \max(ubd|f(x)|, ubd|g(t)|).$$

By the Test, since $g(t)$ is integrable, given $\varepsilon > 0$ there exists a subdivision of $[a, b]$ such that the intervals in which $osc[g(t)] > \varepsilon$ have total length $< \varepsilon$. Call these the *intervals of type* 1. The remaining intervals, in which $osc[g(t)] \leq \varepsilon$, are of two kinds:

Type 2. Those in which $|g(t)| < \varepsilon$ somewhere. Throughout such an interval $|g(t)| < 2\varepsilon$.

Type 3. Those in which $|g(t)| \geq \varepsilon$ everywhere. These we shall further subdivide; let $[u, v]$ be a typical one. Now in $[u, v]$, either $g(t) \geq \varepsilon$ everywhere, or $g(t) \leq -\varepsilon$ everywhere; suppose for example the former. Then by the first mean value theorem for integrals,

$$\frac{G(t'') - G(t')}{t'' - t'} \geq \varepsilon \text{ whenever } u \leq t' < t'' \leq v. \tag{1}$$

By the Test, since $f(x)$ is integrable there exists a subdivision of $G([u, v])$ such that the intervals in which $osc[f(x)] > \varepsilon$ have total length $< \varepsilon^2/N$, where N is the total number of intervals of type 3. Since, by (1), $G(x)$ is strictly increasing in $[u, v]$, the intervals of the subdivision are $[G(\tau_{j-1}), G(\tau_j)]$ $(j = 1, \ldots, m)$, say, where $u = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_m = v$. Also, the oscillation of $f(x)$ in $[G(\tau_{j-1}), G(\tau_j)]$ equals that of $f(G(t))$ in $[\tau_{j-1}, \tau_j]$.

The intervals $[\tau_{j-1}, \tau_j]$ are of two kinds:

Type 3.1. Those in which $osc[f(G(t))] > \varepsilon$; by (1) these have total length $< (1/\varepsilon)(\varepsilon^2/N) = \varepsilon/N$.

Type 3.2. Those in which $osc[f(G(t))] \leq \varepsilon$.

Subdividing each interval of type 3 in the above fashion, we obtain a subdivision of $[a, b]$ into intervals of types 1, 2, 3.1, and 3.2, say $a = t_0 \leq t_1 \leq \ldots \leq t_n = b$; and those of types 1 and 3.1 have total length $< \varepsilon + N(\varepsilon/N) = 2\varepsilon$.

By twice applying the first mean value theorem, we can write

$$\int_{G(a)}^{G(b)} f(x)\,dx = \sum_{i=1}^{n} \int_{G(t_{i-1})}^{G(t_i)} f(x)\,dx$$
$$= \sum_{i=1}^{n} [G(t_i) - G(t_{i-1})]\lambda_i$$
$$= \sum_{i=1}^{n} (t_i - t_{i-1})\lambda_i\mu_i,$$

where λ_i is between the lower and upper bounds of $f(x)$ on $[G(t_{i-1}), G(t_i)]$, and *a fortiori* between the bounds of $f(G(t))$ on $[t_{i-1}, t_i]$, and where μ_i is between the bounds of $g(t)$ on $[t_{i-1}, t_i]$.

Consequently, if $t_{i-1} \leq \xi_i \leq t_i$ for $i = 1, 2, \ldots, n$,

$$\left|\int_{G(a)}^{G(b)} f(x)\,dx - \sum_{i=1}^{n} (t_i - t_{i-1}) f(G(\xi_i))g(\xi_i)\right|$$
$$\leq \sum_{i=1}^{n} (t_i - t_{i-1}) \,.\, |\lambda_i\mu_i - f(G(\xi_i))g(\xi_i)|; \qquad (2)$$

we shall show that this sum is small.

The intervals of types 1 and 3.1 have total length $< 2\varepsilon$, and therefore their contribution to the sum (2) is less than $2\varepsilon \,.\, 2M^2$.

Throughout an interval $[t_{i-1}, t_i]$ of type 2, we have $|g(t)| < 2\varepsilon$, and so also $|\mu_i| \leq 2\varepsilon$. Therefore the total contribution to (2) from the intervals of type 2 is at most $(b - a) \,.\, 4\varepsilon M$.

Finally, let $[t_{i-1}, t_i]$ be an interval of type 3.2. Here we have both $osc[g(t)] \leq \varepsilon$ and $osc[f(G(t))] \leq \varepsilon$.

Consequently,

$$|\lambda_i\mu_i - f(G(\xi_i))g(\xi_i)| = |\lambda_i[\mu_i - g(\xi_i)] + [\lambda_i - f(G(\xi_i))]g(\xi_i)|$$
$$\leq |\lambda_i| \,.\, |\mu_i - g(\xi_i)| + |\lambda_i - f(G(\xi_i))| \,.\, |g(\xi_i)|$$
$$\leq 2\varepsilon M.$$

Therefore the total contribution to (2) from the intervals of type 3.2 is at most $(b - a)2\varepsilon M$.

Combining these results, we have

$$\left|\int_{G(a)}^{G(b)} f(x)\,dx - \sum_{i=1}^{n} (t_i - t_{i-1}) f(G(\xi_i))g(\xi_i)\right| < 4\varepsilon M^2 + 6(b - a)\varepsilon M$$

Since this is arbitrarily small, and the ξ_i are arbitrary values in the relevant intervals, the conclusion of the Theorem follows.

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TWO HEXAGONAL DESIGNS

BY PRAKASH CHANDRA SHARMA

These designs were prompted by the "pursuit curves" of Mr. I. J. Good*. The first one consists of his six 'hyperboloids' and the second an equal number of 'spearheads'. The former possesses what may be termed cyclical symmetry and the latter triple axial symmetry.

There seems to be a technical improvement in these designs over those of Mr. Good's insofar as the black areas at the centre of his triangular doodles are conspicuous by their absence over here.

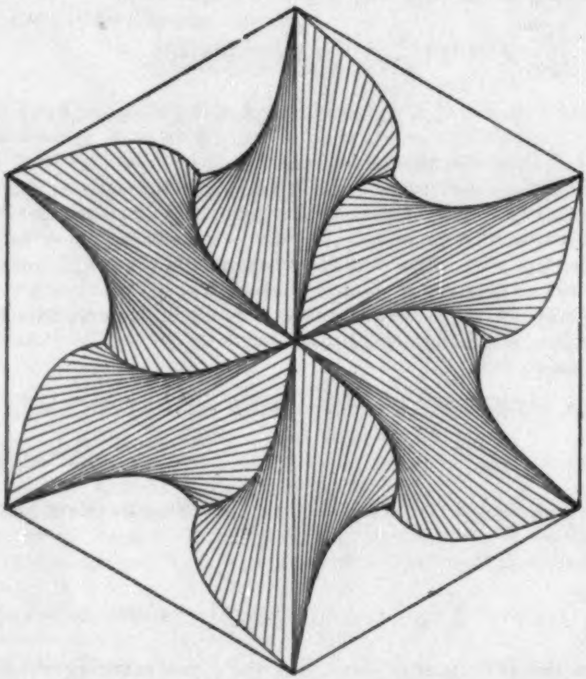

FIG. 1

By superimposition of the first design over the second we obtain a beautiful design consisting of, in addition to six hyperboloids, four three-leaved flowers of which the middle one has latticed interstices.

* I. J. Good, 'Pursuit Curves and Mathematical Art', The Mathematical Gazette, vol. XLIII No. 343 (Feb. 1959), page 34.

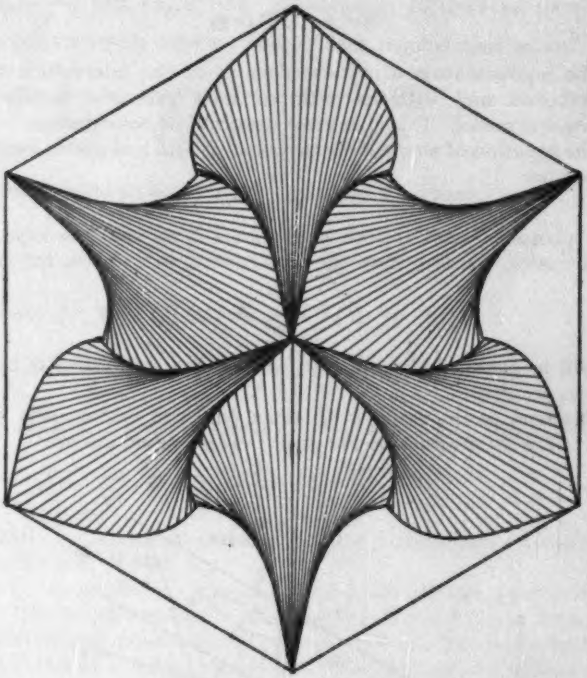

FIG. 2

Again by superimposition of either the first design over its 'inverse' or of the second design over its inverse by *one shift* we get yet another more beautiful design consisting of six latticed 'evolutes of an ellipse' together with a six-leaved flower in the centre.

In passing it may be interesting to note that *all* the curved lines of these designs—basic and compounded ones—are of equal lengths.

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1956. Uniform cross section.

5th Division Form II. "What do you think is meant by a post of uniform cross section?"

After a long silence one hand goes up "A parcel containing school uniform."—[Per Miss E. M. Busbridge.]

A MODEL OF A TWISTED CUBIC

By A. J. Bayes

The representation of a twisted cubic as the intersection of a hyperboloid and cylinder with common generator makes an interesting model. This note gives details of its construction.

The equation of a circular hyperboloid Q with real generators can be written

$$(x-a)^2 + y^2 - c^2z^2 = 1.$$

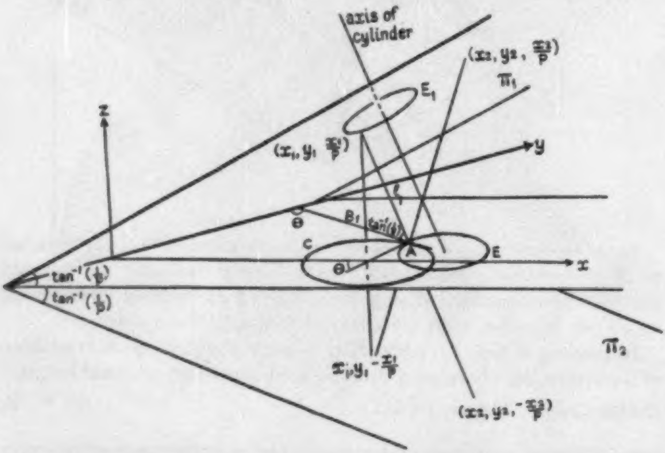
The planes π_1 and π_2 with equations $x = pz$ and $x = -pz$ cut Q in conics, whose orthogonal projection onto $z = 0$ is the conic

$$(x-a)^2 + y^2 - \frac{c^2}{p^2}x^2 = 1 \qquad z = 0 \tag{1}$$

It will be assumed that $|p| > |c|$, so that the sections of Q by π_1 and π_2 are ellipses.

The lines of the reguli in Q, when projected orthogonally onto $z = 0$, are the tangents to the circle C whose equations are

$$(x-a)^2 + y^2 = 1 \qquad z = 0 \tag{2}$$



Each tangent is the projection of two lines, one from each regulus. Let the tangent of C at $A \equiv (a + \cos\theta, \sin\theta, 0)$ meet the ellipse (1) in $B_1 \equiv (x_1, y_1, 0)$ and $B_2 \equiv (x_2, y_2, 0)$, where $x_1 \leqslant x_2$. Then this tangent is the projection of the line l passing through $\left(x_1, y_1, \dfrac{x_1}{p}\right)$ and $\left(x_2, y_2, \dfrac{-x_2}{p}\right)$, and of the line l' passing through

$\left(x_1, y_1, \frac{-x_1}{p}\right)$ and $\left(x_2, y_2, \frac{x_2}{p}\right)$. By choosing different values of θ the points at which different lines of each regulus meet π_1 and π_2 can be found.

Let the distance between A and B_1 be b_1. Then

$$x_1 = a + \cos\theta - b_1 \sin\theta \tag{3}$$

The line l makes an angle $\tan^{-1}\frac{1}{c}$ with $z = 0$. Hence

$$\frac{x_1}{pb_1} = \tan\left(\tan^{-1}\frac{1}{c}\right) = \frac{1}{c} \tag{4}$$

Eliminating x_1 from (3) and (4)

$$b_1 = \frac{c(a + \cos\theta)}{p + c\sin\theta} \tag{5}$$

Similarly, if b_2 is the (absolute) distance between A and B_2 then

$$b_2 = \frac{c(a + \cos\theta)}{p - c\sin\theta} \tag{6}$$

In practice equations (5) and (6) provide a good method for finding the positions of B_1 and B_2.

Let l_1, the generator common to Q and the cylinder, meet $z = 0$ in A_1. The cylinder meets $z = 0$ in an ellipse E which passes through A_1. The twisted cubic touches l_1 at a point D and it makes a better model if D is at infinity. Since Q is circular this can be achieved by choosing E so that it makes a right angle with C at A_1. The axis of the cylinder is parallel to A_1 and passes through the centre of E, so that when E is chosen the shape and position of the cylinder are defined. The generator through any point of E meets π_1 and π_2 in points whose coordinates can be found by elementary trigonometry. By choosing five such points the ellipses E_1, E_2 which π_1, π_2 cut on the cylinder can be found.

A simpler method can be used when the cylinder is circular. E_1 then has the following properties.

1. Its minor axis has the same length as the minor axis of E.
2. $\dfrac{\text{length of minor axis}}{\text{length of major axis}} = \sin[\angle(\pi_1, l_1)]$.
3. Its major axis is parallel to the orthogonal projection of l_1 on π_1.
4. Its centre is the point where the axis of the cylinder meets π_1.

These properties enable E_1 easily to be found by trial and error methods, to an accuracy sufficient for making the model. The purist will be able to find the relevant equations.

If the scale along the y axis is changed then Q is no longer circular; thus the construction is quite general.

The model is formed by strings which pass between π_1 and π_2 and represent the cylinder and the two reguli of the hyperboloid. The line l_1 is shown and can be distinguished by colour, and the twisted cubic is picked out by beads on the strings of the cylinder.

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A GENERALISATION OF SIMSON'S THEOREM

BY S. ZYLBERTREST

Let ω be a point and $ABCDEF$ a polygon, convex or not, which we call (P). Projecting ω orthogonally on the sides AB, BC, CD, DE, EF, FA of (P), and joining the respective projections, we obtain the polygon $A_1B_1C_1D_1E_1F_1$, called (P_1) [see Figure 1]. We call (P_1) the *first pedal* of ω with respect to (P).

By projecting ω orthogonally on the sides A_1B_1, B_1C_1, C_1D_1, D_1E_1, E_1F_1, F_1A_1 of (P_1) and joining the projections, we obtain the polygon $A_2B_2C_2D_2E_2F_2$, or (P_2), the *second pedal* of ω with respect to (P).

The method of forming the third, fourth, and in general the nth pedal of ω with respect to (P) is now obvious; the nth pedal is called (P_n).

THEOREM. *The pedal (P_{n-2}) of a point ω, lying on a circle, with respect to a polygon (P) of n sides inscribed in the circle, is a straight line.*

If $n = 3$, (P) is a triangle, and (P_{n-2}) is the Simson line. The theorem and its converse will be proved first in the case where $n = 6$, so that (P) is a hexagon. The generalisations for any n will follow.

Let (Γ) be a circle and ω a point on (Γ) [see figure 1],
- (P) be a hexagon $ABCDEF$ inscribed in (Γ),
- (P_1) be the first polar $A_1B_1C_1D_1E_1F_1$ of ω with respect to (P),
- (P_2) be the second polar $A_2B_2C_2D_2E_2F_2$ of ω with respect to (P),
- (P_3) be the third polar $A_3B_3C_3D_3E_3F_3$ of ω with respect to (P),
- (P_4) be the fourth polar $A_4B_4C_4D_4E_4F_4$ of ω with respect to (P).

We must prove that A_4, B_4, C_4, D_4, E_4, F_4 are collinear. We prove that any three of these points, A_4, B_4, C_4 for example, are collinear.

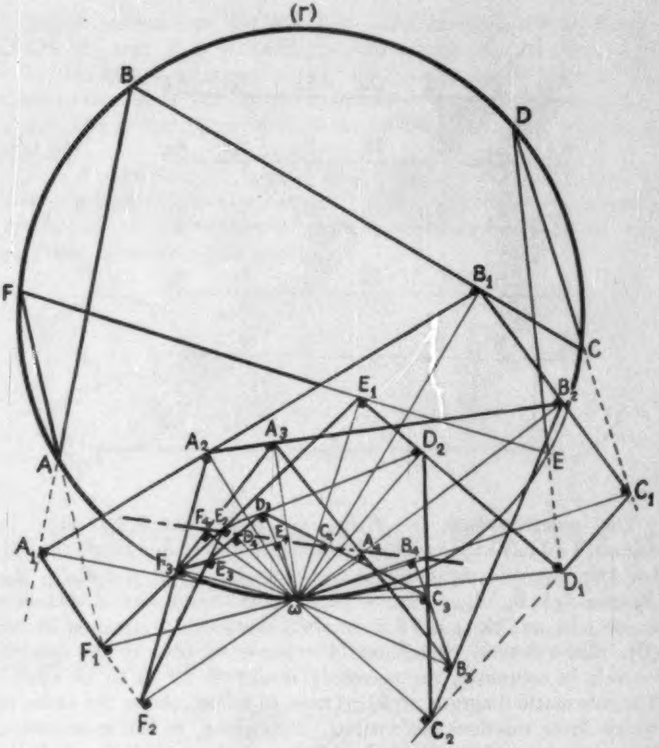
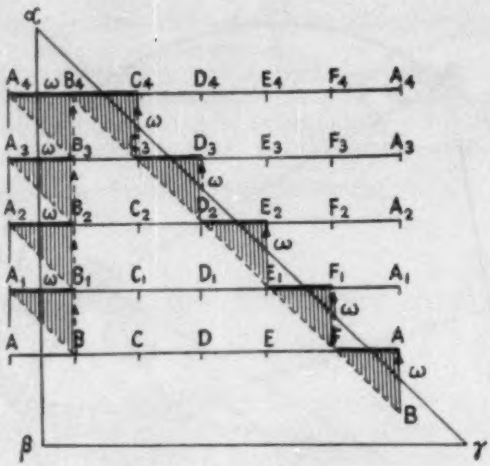


FIG. 1

I	II
$\omega A_4B_3B_4$	$(A_4B_4, A_3B_3) = (\omega B_4, \omega B_3) + k\pi$
$\omega A_3B_2B_3$	$(A_3B_3, A_2B_2) = (\omega B_3, \omega B_2) + k\pi$
$\omega A_2B_1B_2$	$(A_2B_2, A_1B_1) = (\omega B_2, \omega B_1) + k\pi$
ωA_1BB_1	$(A_1B_1, AB) = (\omega B_1, \omega B) + k\pi$
ωAFB	$(AB, AF) = (\omega B, \omega F) + k\pi$
ωF_1E_1F	$(AF, F_1E_1) = (\omega F, \omega E_1) + k\pi$
$\omega E_2D_2E_1$	$(F_1E_1, E_2D_2) = (\omega E_1, \omega D_2) + k\pi$
$\omega D_3C_3D_2$	$(E_2D_2, D_3C_3) = (\omega D_2, C_3) + k\pi$
$\omega C_4B_4C_3$	$(D_3C_3, C_4B_4) = (\omega C_3, B_4) + k\pi$
By addition	$(A_4B_4, C_4B_4) = (\omega B_4, \omega B_4) + k\pi$
	$= k\pi$



The quadrilaterals $\omega A_4B_3B_4$, $\omega A_3B_2B_3$, $\omega A_2B_1B_2$, etc., in column I all have the point ω in common, and each, except ωAFB, has two opposite right angles (such as A_4, B_4 in $\omega A_4B_3B_4$, A_3, B_3 in $\omega A_3B_2B_3$, A_2, B_2 in $\omega A_2B_1B_2$, and so on), and is therefore cyclic. In addition, ωAFB is cyclic because its vertices lie on (Γ). The relations in column II express, for each of the quadrilaterals in column I, the necessary condition for it to be cyclic. The schematic diagram, which is easy to follow, shows the order in which these relations are formed. Describing, in the same order, the perimeters of the shaded triangles (such as $A_4B_3B_4$, $A_3B_2B_3$, $A_2B_1B_2$, and so on), we form the quadrilaterals $\omega A_4B_3B_4$, $\omega A_3B_2B_3$, $\omega A_2B_1B_2$, and so on, in column I. For the direct theorem we describe the diagram in the order $\alpha\beta\gamma\alpha$, for the converse theorem in the order $\beta\alpha\gamma$.

This result proves that the three consecutive points A_4, B_4, C_4 are collinear. Since these were arbitrarily chosen from the six points A_4, B_4, C_4, D_4, E_4, F_4, it follows that any three consecutive points of the set are collinear, and therefore that all six are collinear.

Note 1. The above proof seems to break down if two adjacent points, A_4, B_4 for example coincide. For the line A_4B_4 seems to be indeterminate, and it therefore makes no sense to say that C_4 lies on A_4B_4.

We return to figure 1, and examine the cyclic quadrilateral $\omega A_4B_3B_4$. The circumscribing circle has ωB_3 as diameter. Let β_3 be the centre of this circle, which we call (β_3).

In the general case the line A_4B_4 cuts the circle (β_3) at distinct points A_4 and B_4. When B_4 concides with A_4, the line A_4B_4 coincides with the tangent to (β_3) at A_4, and its direction is thus determined: it is that of the perpendicular at A_4 to the radius β_3A_4. The preceding proof therefore still applies when A_4 and B_4 coincide.

Note 2. The theorem applies to all polygons (P) whose vertices and ω belong to the same circle (Γ). In particular, two or more vertices of (P) and thus two or more sides, adjacent or not, can coincide. Here are some examples.

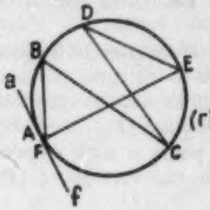


FIG. 2

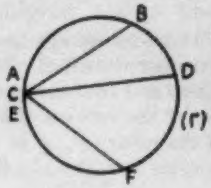


FIG. 3

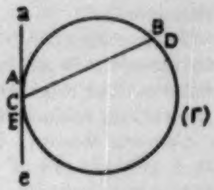


FIG. 4

FIG. 2. In the hexagon *ABCDEF* two adjacent vertices *A* and *F* have coincided; the side *AF* coincides with the tangent at *A*.

FIG. 3. In the hexagon *ABCDEF* three pairs of sides, *AB* & *BC*, *CD* & *DE*, *EF* & *FA*, have coincided, and therefore the 3 vertices *A*, *C*, *E* have coincided.

FIG. 4. In the pentagon *ABCDE* four sides have coincided into one, and therefore the vertices *A*, *C*, *E* have coincided and *B* and *D* have coincided. The 5th side is the tangent *ae*.

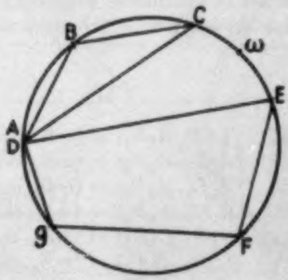


FIG. 5

We now examine the case of the heptagon *ABCDEFG* represented in figure 5. The vertex D has coincided with A. This separates the heptagon into two closed contours, *ABCA* and *DEFGD*. The first is a triangle, the second a quadrilateral: we call the latter (p).

The pedal (P_1) of ω with respect to the heptagon (P) will have its first three vertices A_1, B_1, C_1, collinear (by Simson's theorem).

It follows that the vertices A_2 and B_2 of (P_2) coincide. This does not prevent A_2B_2 from having a determinate direction, which is that of the tangent at A_2 to the circle (β_1) with diameter ωB_1 (see note 1).

(In general, if the vertices A_k, B_k of any pedal (P_k) coincide, the line A_kB_k coincides with the tangent at A_k to the circle (β_{k-1}) with diameter ωB_{k-1}).

In turn, the pedal (P_2) will have D_2, E_2, F_2 collinear, since these are three vertices of the pedal (p_2) of ω with respect to (p). It follows that the vertices D_3 and E_3 of (P_3) coincide, and the line D_3E_3 coincides with the tangent at D_3 to the circle (ξ_2) with diameter ωE_2.

Each coincidence of two vertices will have consequences similar to these which we have just described. Now a polygon (P) can have several coincidences, and the reader will be able to discover the various collinearities of the vertices of any polar (P_k).

Converse theorem. If the polar (P_{n-2}) of a point ω with respect to a polygon (P) of n sides is a straight line, the vertices of (P) and ω are concyclic.

In the particular case where $n = 6$, so that (P) is a hexagon $ABCDEF$, the vertices A_4, B_4, C_4, D_4, E_4, F_4 of (P_4) are collinear, by hypothesis. We need to prove that A, B, C, D, E, F and ω are concyclic.

We return to the schematic diagram, which we used to illustrate the direct theorem, and follow the route $\beta\alpha\gamma$, to establish the results of column II. The diagram also enables us, by describing the shaded triangles, to fill in column I, indicating the cyclic quadrilaterals with common vertex ω to which these relations apply.

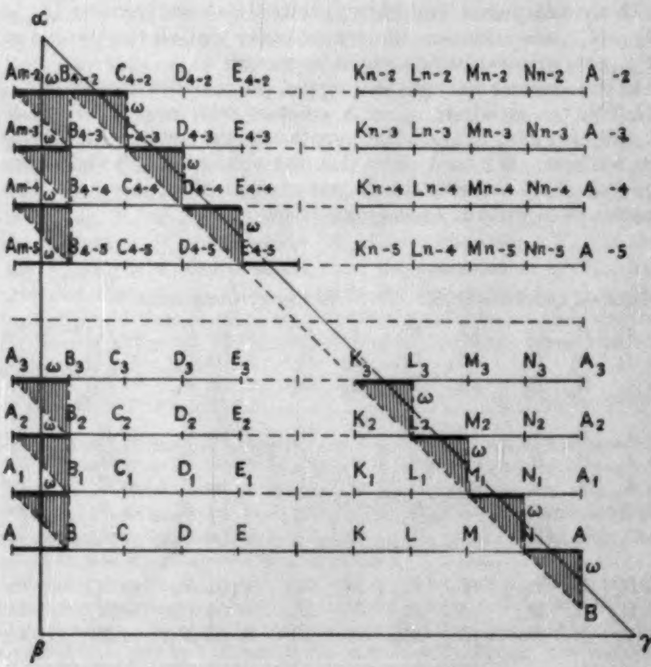
I	II
$\omega\, A_1\, B\, B_1$	$(AB, A_1B_1) = (\omega B, \omega B_1) + k\pi$
$\omega\, A_2\, B_1\, B_2$	$(A_1B_1, A_2B_2) = (\omega B_1, \omega B_2) + k\pi$
$\omega\, A_3\, B_2\, B_3$	$(A_2B_2, A_3B_3) = (\omega B_2, \omega B_3) + k\pi$
$\omega\, A_4\, B_3\, B_4$	$(A_3B_3, A_4B_4) = (\omega B_3, \omega B_4) + k\pi$
We have also,	$(A_4B_4, C_4B_4) = (\omega B_4, \omega B_4) + k\pi$
$\omega\, C_4\, B_4\, C_3$	$(C_4B_4, D_3C_3) = (\omega B_4, \omega C_3) + k\pi$
$\omega\, D_3\, C_3\, D_2$	$(D_3C_3, E_2D_2) = (\omega C_3, \omega D_2) + k\pi$
$\omega\, E_2\, D_2\, E_1$	$(E_2D_2, F_1E_1) = (\omega D_2, \omega E_1) + k\pi$
$\omega\, F_1\, E_1\, F$	$(F_1E_1, AF) = (\omega E_1, \omega F) + k\pi$

By addition $(AB, AF) = (\omega B, \omega F) + k\pi$

This result proves that the quadrilateral ωFAB is cyclic. A cyclic permutation enables us to extend this property to the

quadrilaterals ωABC, ωBCD, ωCDE, etc. Now the circles ωABC, ωBCD, ωCDE, etc., coincide, since each has three points in common with the next. Hence A, B, C, D, E, F and ω are concyclic.

Having proved the theorem and its converse in the particular case where $n = 6$, we can pass to the proof of the general case. For the direct theorem, let (Γ) be a circle, ω a point on (Γ), (P) a polygon $ABCDE \dots KLMN$ of n sides inscribed in (Γ), and (P_{n-2}) the $(n-2)$th pedal $A_{n-2}B_{n-2} \dots M_{n-2}N_{n-2}$ of ω with respect to (P). We must prove that A_{n-2}, B_{n-2}, $\dots$, M_{n-2}, N_{n-2} are collinear. Now three consecutive vertices of (P_{n-2}), for example A_{n-2}, B_{n-2}, C_{n-2}, are collinear. This is proved by considering the table and schematic diagram below.[e] The order in which the conditions are established is given by describing the diagram in the order $\alpha\beta\gamma\alpha$. The inscribed quadrilaterals in the column on the left are derived from the perimeters of the shaded triangles in the diagram.



Inscribed Quadrilaterals	Necessary Conditions
$\omega\, A_{n-2}B_{n-3}B_{n-2}$	$(A_{n-2}B_{n-2}, A_{n-3}B_{n-3}) = (\omega\, B_{n-2}, \omega\, B_{n-3}) + k\pi$
$\omega\, A_{n-3}B_{n-4}B_{n-3}$	$(A_{n-3}B_{n-3}, A_{n-4}B_{n-4}) = (\omega\, B_{n-3}, \omega\, B_{n-4}) + k\pi$
$\cdots$	$\cdots$
$\omega\, A_2 \; B_1 \; B_2$	$(A_2 \; B_2, \; A_1 \; B_1) = (\omega\, B_2, \; \omega\, B_1) + k\pi$
$\omega\, A_1 \; B \; B_1$	$(A_1 \; B_1, \; A \; B) = (\omega\, B_1, \; \omega\, B) + k\pi$
$\omega\, A \; N \; B$	$(A \; B \; A \; N) = (\omega\, B, \; \omega\, N) + k\pi$
$\omega\, N_1 \; M_1 \; N$	$(A \; N, \; N_1 \; M_1) = (\omega\, N, \; \omega\, M_1) + k\pi$
$\omega\, M_2 \; L_2 \; M_1$	$(N_1 \; M_1, \; M_2 \; L_2) = (\omega\, M_1, \; \omega\, L_2) + k\pi$
$\cdots$	$\cdots$
$\omega\, D_{n-3}C_{n-3}D_{n-4}$	$(E_{n-4}D_{n-4}, D_{n-3}C_{n-3}) = (\omega\, D_{n-4}, \omega\, C_{n-3}) + k\pi$
$\omega\, C_{n-2}B_{n-2}C_{n-3}$	$(D_{n-3}C_{n-3}, C_{n-2}B_{n-2}) = (\omega\, C_{n-3}, \omega\, B_{n-2}) + k\pi$
By addition	$(A_{n-2}B_{n-2}, C_{n-2}B_{n-2}) = (\omega\, B_{n-2}, \omega\, B_{n-2}) + k\pi = k\pi$

This result proves that three arbitrarily chosen vertices, A_{n-2}, B_{n-2}, C_{n-2} are collinear. We deduce easily that all the vertices of (P_{n-2}) are collinear, which was to be proved.

In the converse theorem, the polygon (P) of n sides is $ABCDE \ldots KLMN$, the pedal (P_{n-2}) of a point ω with respect to (P) is $A_{n-2}B_{n-2} \ldots M_{n-2}N_{n-2}$. By hypothesis, the vertices of (P_{n-2}) are collinear. We must show that the vertices of (P) and ω are concyclic. We describe the diagram now in the order $\beta\alpha\gamma$. This enables us to write down the table below.

Inscribed Quadrilaterals	Necessary Conditions
$\omega\, A_1 \; B \; B_1$	$(A \; B, \; A_1 \; B_1) = (\omega\, B, \; \omega\, B_1) + k\pi$
$\omega\, A_2 \; B_1 \; B_2$	$(A_1 \; B_1, \; A_2 \; B_2) = (\omega\, B_1, \; \omega\, B_2) + k\pi$
$\omega\, A_3 \; B_2 \; B_3$	$(A_2 \; B_2, \; A_3 \; B_3) = (\omega\, B_2, \; \omega\, B_3) + k\pi$
$\cdots$	$\cdots$
$\omega\, A_{n-4}B_{n-5}B_{n-4}$	$(A_{n-5}B_{n-5}, A_{n-4}B_{n-4}) = (\omega\, B_{n-5}, \omega\, B_{n-4}) + k\pi$
$\omega\, A_{n-3}B_{n-4}B_{n-3}$	$(A_{n-4}B_{n-4}, A_{n-3}B_{n-3}) = (\omega\, B_{n-4}, \omega\, B_{n-3}) + k\pi$
$\omega\, A_{n-2}B_{n-3}B_{n-2}$	$(A_{n-3}B_{n-3}, A_{n-2}B_{n-2}) = (\omega\, B_{n-3}, \omega\, B_{n-2}) + k\pi$
By hypothesis,	$(A_{n-2}B_{n-2}, C_{n-2}B_{n-2}) = (\omega\, B_{n-2}, \omega\, B_{n-2}) + k\pi$
$\omega\, C_{n-2}B_{n-2}C_{n-3}$	$(C_{n-2}B_{n-2}, D_{n-3}C_{n-3}) = (\omega\, B_{n-2}, \omega\, C_{n-3}) + k\pi$
$\cdots$	$\cdots$
$\omega\, L_3 \; K_3 \; L_2$	$(L_3 \; K_3, \; M_2 \; L_2) = (\omega\, K_3, \; \omega\, L_2) + k\pi$
$\omega\, M_2 \; L_2 \; M_1$	$(M_2 \; L_2, \; N_1 \; M_1) = (\omega\, L_2, \; \omega\, M_1) + k\pi$
$\omega\, N_1 \; M_1 \; N$	$(N_1 \; M_1, \; A \; N) = (\omega\, M_1, \; \omega\, N) + k\pi$
By addition	$(A \; B, \; A \; N) = (\omega\, B, \; \omega\, N) + k\pi$

The relation of the last line proves that the circle NAB passes through ω. A cyclic permutation shows that the same is true for the circles ABC, BCD, CDE and so on. Now each consecutive pair of these circles has three common points (such as ω, B, C for the circles ωABC, ωBCD), so there is one circle which contains all the vertices of the polygon $ABCDE \ldots KLMN$ and ω, which was to be proved.

The reader may wish to verify the converse theorem experimentally, by reconstructing the polygon (P), starting with a point ω and a pedal (P_{n-2}) whose n vertices are all collinear. There is no difficulty in constructing (P_{n-3}), if the vertices of (P_{n-2}) are distinct. If these vertices are A_{n-2}, $B_{n-2}, \ldots$, the perpendicular at A_{n-2} to ωA_{n-2} meets the perpendicular at B_{n-2} to ωB_{n-2} at B_{n-3}. Similarly, the perpendicular at B_{n-2} to ωB_{n-2} meets the perpendicular at C_{n-2} to ωC_{n-2} at C_{n-3}, and so on. Having constructed (P_{n-3}), we construct (P_{n-4}), and continuing in this way we arrive at a polygon (P) whose vertices are concyclic with ω.

Suppose now that, before reaching (P), the construction leads us to a pedal (P_k) $(1 \leqslant k \leqslant n-2)$, of which two consecutive vertices, A_k and B_k for example, coincide. The construction given above for B_{k-1} now fails, since the perpendiculars which should give B_{k-1} now coincide. However, we draw the reader's attention to the fact, already stated in Note 1, that the direction of the line A_kB_k is not indeterminate, even though A_k and B_k coincide. This direction is that of the tangent at A_k to the circle (B_{k-1}) with diameter ωB_{k-1}. If we refer to figure 1, we establish the collinearity of A_1, A_2, B_1, or A_2, A_3, B_2, or A_3, A_4, B_3, and in general, A_k, A_{k+1}, A_k. Since the tangent A_kB_k to the circle (B_{k-1}) with diameter ωB_{k-1} is the line $A_{k+1}B_k$, it is easy to draw, since the construction of A_{k+1} has preceded that of A_k. We construct the perpendicular (π) at A_k to A_kB_k, and then the line (π_1) which is parallel to (π) and twice as far from ω as (π) is. The intersection of (π_1) with the perpendicular at A_k to ωA_k gives B_{k-1}.

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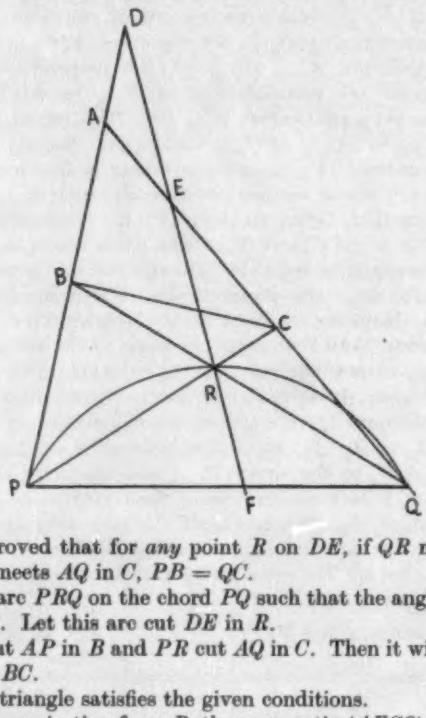
1957. "Three hundred smokers loyal to one of three major brands of cigarette were given the three brands to smoke (with labels taped) and asked to identify their own favourite brand.

Result: thirty-five per cent were able to do so; and under the law of averages pure guesses would have accounted for a third of the correct identifications. In short, something less than two per cent could be credited with any real power of discrimination."—From "The Hidden Persuaders," by Vance Packard, Penguin Edition, p. 45. [Per Mr. A. G. Sillitto.]

2937. A triangle construction

To construct a triangle, given two sums of sides $(a + b)$, $(a + c)$, and the angle A.

Construction: Draw lines AP, AQ such that $\angle PAQ$ is the given $\angle A$, $AP = (a + c)$, and $AQ = (a + b)$. On PA mark D such that $PD = AQ$. On QA mark E such that $QE = AP$. Join DE.



It will be proved that for *any* point R on DE, if QR meets AP in B and PR meets AQ in C, $PB = QC$.

Draw an arc PRQ on the chord PQ such that the angle in the arc is $90^\circ + \frac{1}{2}A$. Let this arc cut DE in R.

Let QR cut AP in B and PR cut AQ in C. Then it will be proved that $PB = BC$.

Thus the triangle satisfies the given conditions.

Proof. By projection from P, the cross ratio $(AECQ)$ = the cross ratio $(DERF)$.

By projection from Q, c.r.$(DERF)$ = c.r.$(DABP)$.

Therefore c.r. $(AECQ)$ = c.r. $(DABP)$.

But $\left.\begin{array}{l} AQ = DP \\ EQ = AP \end{array}\right\}$ by construction,

and so $CQ = BP$.

Suppose $BC > BP.$

Then $\angle BPC > \angle BCP.$

Also $\angle BPC + \angle BCP = \angle ABC$

and so $\angle BCP < \frac{1}{2}\angle ABC.$

Similarly $\angle CBQ < \frac{1}{2}\angle ACB.$

Therefore $\angle BRC > 180^\circ - \frac{1}{2}\angle ABC - \frac{1}{2}\angle ACB$

i.e. $\angle BRC > 90^\circ + \frac{1}{2}\angle A.$

Since we made $\angle BRC$ equal to $90^\circ + \frac{1}{2}\angle A$, the supposition cannot be true. Similarly we can prove that the supposition $BC < BP$ is untenable.

Hence $BC = BP$, which completes the proof.

The use of cross ratios, a projective tool, seems rather out of character in a problem of this nature, but I have been unable to find any simpler way of proving (1).

The problem of constructing a triangle given $(a + b)$, $(b + c)$ and $\angle A$ has a similar solution.

T. E. EASTERFIELD

2938. On note 2921

1. Morley's conjecture in Note 2921 that if $2^n - 1 = p$ is prime then $2^p - 1$ is also prime is false. The electronic computer in Urbana Illinois showed that although $2^{13} - 1 = 8191$ is prime, $2^{8191} - 1$ is composite; the computer took about 40 hours to show this, and as far as I know the result was not checked.

Budapest PAUL ERDÖS

2. Some of the numbers in G. H. Morley's conjecture were tested in Toronto on the new IBM 704 Data Processing System, with the following results.

657,710,813 is prime.

$1{,}161{,}737{,}179 = 1559 \times 745181$

2,147,483,647 is prime.

For each number the initial programming took less than an hour, and the machine time was less than 5 minutes.

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3. Morley's conjecture that $2^p - 1$ is prime if $p = 2^n - 1$ is prime was proposed by E. Catalan (Mélanges Math. Bruxelles, 1 (1885), p. 147. Cf. L. E. Dickson, History of the Theory of Numbers,

Vol. 1, p. 24), and is known to fail for $n = 13$, $n = 17$, $n = 19$. Failure in the case $n = 13$ was shown by D. J. Wheeler in 1953 (see R. M. Robinson, "Mersenne and Fermat numbers," *Proc. Amer. Math. Soc.* 5 (1954), p. 842–846). With $n = 17$, $2^p - 1$ has the factor $1768\ (2^{17} - 1) + 1$, and with $n = 19$, $2^p - 1$ has the factor $120\ (2^{19} - 1) + 1$; these factorisations were found by R. M. Robinson "Some factorisations of numbers of the form $2^n \pm 1$," *Math. Tables* and other aids to computation, 11 (1957), p. 265–268.

Bagatela 15 *m*. 49, *Warsaw* 10, *Poland*. A. Makowski

4. The hypothesis put forward by G. H. Morley is not in fact correct, since the number 83,828,316,391 is divisible by 53.

A reason for there being so many cases in which the hypothesis is verified is that numbers of the form $(x^p - a^p)/(x - a)$, with x prime and prime to a are considerably restricted as to their possible factors—any factor must be of the form $2kp + 1$, where k is an integer. To prove this let $q(>p)$ be a prime dividing $(x^p - a^p)/(x - a)$ and, working modulo q, let b be such that $x \equiv ab$. Then $(ab)^p \equiv a^p$ whence $b^p \equiv 1$ if q is prime to a. By Fermat's Theorem $b^{q-1} \equiv 1$ and so p divides $q - 1$. Hence $q - 1 = mp$ and, since q is odd, m is even.

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5. 657,710,813 is prime,

$$1{,}161{,}737{,}179 = 1{,}559 \times 745{,}181,$$

$$83{,}828{,}316{,}391 = 53 \times 79 \times 20{,}021{,}093,$$

3,835,856,903,971 is prime.

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2939. On the representation of numbers as sums of triangular numbers

Numbers of the form $\frac{1}{2}n(n + 1)$, $n \geq 0$, are called triangular numbers. Theorem 1 below asserts that every non-negative integer is expressible as the sum of three triangular numbers and theorem 2 gives the general form of all numbers not expressible as the sum of two triangular numbers. M. Satyanarayana [1] proved recently that no Fermat number $F_n = (2^{2^n} + 1)$, $n \geq 1$, is triangular and that there are infinitely many Mersenne's numbers $M_n = (2^n - 1)$, $n \geq 1$, which are not triangular. Naturally one is led to examine whether any of the above numbers is expressible as the sum of two triangular numbers. Theorems 3 and 4 below deal with these aspects.

The results of the paper are—

Theorem 1: Every non-negative integer n is expressible as the sum of three triangular numbers;

Theorem 2: Numbers of the form $\frac{p^{2k+1}s - 1}{4}$, where p is a prime of the form $4n + 3$ znd s is a number of the form $4n + 3$ and prime to p, are not expressible as the sum of two triangular numbers; all other numbers are expressible as sums of two triangular numbers;

Theorem 3: No Fermat Number $F_n (n > 1)$ is expressible as the sum of two triangular numbers;

Theorem 4: There are infinitely many Mersenne's numbers not expressible as the sum of two triangular numbers.

Before proceeding with the proofs, we recall the well-known and easily proved result that a necessary and sufficient condition for t to be triangular is that $(1 + 8t)$ is a perfect square ... (0.1)

Proof of Theorem 1:

Let n be non-negative and consider $(3 + 8n)$. Because this is not of the form $4^k(8m + 7)$, by a well-known result [2], it is expressible as the sum of three squares. So, let

$$(3 + 8n) = x^2 + y^2 + z^2.$$

Since the left hand side is $\equiv 3 \pmod{8}$, each of x, y and z is odd. Writing them as $(2x' + 1)$, $(2y' + 1)$, $(3z' + 1)$ respectively, we have that

$$8n = \{(2x' + 1)^2 - 1\} + \{(2y' + 1)^2 - 1\} + \{(2z' + 1)^2 - 1\},$$

giving

$$n = \tfrac{1}{2}x'(x' + 1) + \tfrac{1}{2}y'(y' + 1) + \tfrac{1}{2}z'(z' + 1).$$

Hence the theorem.

Proof of Theorem 2:

If possible, let $n = \frac{p^{2k+1}s - 1}{4}$ be expressible as the sum of two triangular numbers x and y. Then

$$(2 + 8n) = 2p^{2k+1}s = (1 + 8x) + (1 + 8y).$$

Now by virtue of (0.1), $(1 + 8x)$ and $(1 + 8y)$ are perfect squares, so that we can write

$$(2 + 8n) = \alpha^2 + \beta^2, \ \alpha \text{ and } \beta \text{ being odd.}$$

Hence we have that

$$(1 + 4n) = \left(\frac{\alpha + \beta}{2}\right)^2 + \left(\frac{\alpha - \beta}{2}\right)^2. \qquad \ldots . (2.1)$$

But since the number $(4n + 1) = p^{2k+1}s$ has, in its canonical representation, the prime p (of the form $4n + 3$) raised to an odd power, by a well known result [3], it is not expressible as the sum of

two squares, contradicting (2.1). Hence the first part of the theorem follows.

For the second part, let n be a positive integer not in the given form. Then $(4n + 1)$ will not have in its canonical representation any prime of the form $4n + 3$ raised to an odd power. Hence it is expressible as the sum of two squares [3], say

$$(4n + 1) = x^2 + y^2.$$

Obviously, one of x and y is odd and the other is even. Hence

$$(2 + 8n) = (x + y)^2 + (x - y)^2,$$

where both of $(x + y)$ and $(x - y)$ are odd. Denoting them by $(2x' + 1)$ and $(2y' + 1)$ respectively, we have.

$$8n = \{(2x' + 1)^2 - 1\} + \{(2y' + 1)^2 - 1\},$$

giving

$$n = \tfrac{1}{2}x'(x' + 1) + \tfrac{1}{2}y'(y' + 1),$$

proving the second part of the theorem.

Proof of Theorem 3:

Theorem 2 shows that it is enough if we prove that $(4F_n + 1)$ is of the form $p^{2k+1}s$ where p is a prime of the form $4n + 3$ and s is prime to p and is of the form $4n + 3$. We establish this by showing that

$$3|(4F_n + 1) \quad \text{but} \quad 3^2 \nmid (4F_n + 1). \qquad \ldots. (3.1)$$

We can easily verify that 3 divides $(4F_n + 1) = (4 \cdot 2^{2^n} + 5)$. Reducing this modulo 3^2, we have

$$(4 \cdot 2^{2^n} + 5) \equiv 5(1 - 2^{2^n}) \text{ (modulo 9)}.$$

Hence 3^2 divides $(4F_n + 1)$ if and only if

$$2^{2^n} \equiv 1 \text{ (modulo 9)}.$$

Now, observing that 2 is a primitive root of 9, the above would imply that 2^n is divisible by $\phi(9) = 6$, which is false. This proves (3.1) and hence the theorem.

Before proceeding to prove Theorem 4, we prove the following

Lemma: Suppose $p > 3$ is a prime of the form $4n + 3$ and 2 is a primitive root of p^2. Then there are infinitely many n for which

$$p|(4M_n + 1) \quad \text{but} \quad p^2 \nmid (4M_n + 1).$$

Proof. Firstly we observe that primes mentioned in the hypothesis do exist, for 11 is one such.

Now, 3 being prime to p^2 and 2 is a primitive root, there is a μ such that

$$2^\mu \equiv 3 (\text{modulo } p^2) \qquad \ldots. (4.1)$$

We now take

$$n = (p - 1)t + \mu - 2, \text{ where } p \nmid t.$$

With this choice of n we have

$$(4M_n + 1) = (2^{n+2} - 3) = (2^{p-1})^t\, 2^\mu - 3,$$

which, by Fermat's theorem and (4.1) is

$$\equiv 0 \text{ (modulo } p).$$

Also, by (4.1)

$$(4M_n + 1) \equiv 3\{2^{(p-1)t} - 1\} \text{ (modulo } p^2).$$

p being >3, the above is divisible by p^2 if and only if

$$2^{(p-1)t} \equiv 1 \text{ (modulo } p^2).$$

But since 2 is a primitive of p^2, the above would imply that $\varphi(p^2) = p(p-1)$ divides $(p-1)t$, which, by the very choice of t, is false. Hence for the above choice of n we would have

$$p^2 \nmid (4M_n + 1),$$

proving the Lemma.

Proof of Theorem 4:

The Theorem now follows from Theorem 2 and the Lemma.

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2. Edmund Landau: Elementary Number Theory, (Chelsea Publishing Company, New York, N.Y, 1958), Part III, Chap. IV, pp. 162, Th. 178.
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2940. A note on the Möbius function

The aim of this note is to point out a characteristic property of the Möbius function and to show that it is possible to define the Möbius function by means of this property.

The Möbius function $\mu(n)$ is defined as follows:

$$\left.\begin{array}{lll} (1) & \mu(1) = 1 & \\ (2) & \mu(n) = (-1)^k & \text{if } n \text{ is squarefree and has } k \text{ distinct prime factors} \\ (3) & \mu(n) = 0 & \text{if } n \text{ has a squared factor.} \end{array}\right\} \quad \text{(A)}$$

From this it can be deduced that

$$\sum_{d/n} \mu(d) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1. \end{cases} \tag{B}$$

It can be shown that (B) is the characteristic property of the Möbius function; i.e., if any other function $\mu^*(n)$ has the property (B), then $\mu^*(n) \equiv \mu(n)$.

To prove this, we make use of the Möbius inversion formula which states that, if for any functions $f(n)$ and $g(n)$,

$$g(n) = \sum_{d/n} f(d) \text{ holds,}$$

then

$$f(n) = \sum_{d/n} \mu(d)\, g(n/d). \tag{C}$$

Since $\mu^*(n)$ has the property (B), we have

$$\sum_{d/n} \mu^*(d) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1 \end{cases}$$

and if the function $\nu(n)$ is defined as

$$\nu(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1, \end{cases}$$

then

$$\sum_{d/n} \mu^*(d) = \nu(n) \quad \text{for all } n.$$

So by (C),

$$\mu^*(n) = \sum_{d/n} \mu(d)\, \nu(n/d).$$

In the right side of the above, by definition of $\nu(n)$, all the terms in the sum are zero except when $d = n$. Hence

$$\mu^*(n) = \mu(n)\, \nu(1) = \mu(n).$$

Since this is true for any n, $\mu^*(n) \equiv \mu(n)$.

In the above we have made use of the Möbius inversion formula which is based on the definition of the Mobius function as in (A). In the following we shall show that if we dispense with definition (A) and alternately define a function $\mu^*(n)$ by (B), then we can arrive at equations (A), where $\mu(n)$ is replaced by $\mu^*(n)$; i.e. $\mu^*(n)$ is identical with the Möbius function defined in the conventional way. We shall not make use of any mathematical tool except the induction process to prove this.

THEOREM: *Let $\mu^*(n)$ be defined as*

$$\sum_{d/n} \mu^*(d) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1. \end{cases}$$

Then

(1) $\mu^*(1) = 1$

(2) $\mu^*(n) = (-1)^k$ *if n is squarefree and has k distinct prime factors*

(3) $\mu^*(n) = 0$ *if n has a squared factor.*

Proof. Let $n = 1$. Then by definition of $\mu^*(n)$, $\sum_{d/n} \mu^*(d) = 1$. But the term in the left side has one term only viz., $\mu^*(1)$. Hence $\mu^*(1) = 1$.

Let n be a prime p. Then

$$\begin{aligned}\sum_{d/n} \mu^*(d) &= \mu^*(1) + \mu^*(p)\\ &= 0 \text{ by definition of } \mu^*\end{aligned}$$

and so $\mu^*(p) = -1 = (-1)^1$. Let us now show by induction that $\mu^*(n) = (-1)^k$ if $n = p_1p_2 \cdots p_k$, the p's being all distinct. Let us assume that

$$\mu^*(p_1p_2 \cdots p_r) = (-1)^r \text{ for all } r < \lambda \text{ (say).}$$

Let $N = p_1p_2 \cdots p_\lambda$.

Hence $\mu^*(N) = (-1)^\lambda$.

Let $p_{\lambda+1}$ be a prime distinct from all of $p_1, p_2, \dots, p_\lambda$. Then by definition of μ^*, $\sum_{d/Np_{\lambda+1}} \mu^*(d) = 0$. But

$$\begin{aligned}\sum_{d/Nq_{\lambda+1}} \mu^*(d) &= \mu^*(1) + \sum_1^{\lambda+1} \mu^*(p_i) + \cdots\\ &\quad + \Sigma\mu^*(p_{i_1}p_{i_2} \cdots p_{i_\lambda}(+ \mu^*(Np_{\lambda+1})\\ &= 1 + (-1)^1\binom{\lambda+1}{1} + (-1)^2\binom{\lambda+1}{2} + \cdots\\ &\quad + (-1)^\lambda\binom{\lambda+1}{\lambda} + \mu^*(Np_{\lambda+1}) +\end{aligned}$$

by induction hypothesis;

$$= (1 - 1)^{\lambda+1} + \mu^*(Np_{\lambda+1}) - (-1)^{\lambda+1}.$$

Hence $\mu^*(Np_{\lambda+1}) = (-1)^{\lambda+1}$ and $Np_{\lambda+1}$ has $\lambda + 1$ distinct prime factors. This means that if the induction hypothesis is true for all $r < \lambda$, then it is true for all $r < \lambda + 1$; and this is true when $\lambda = 1$. Hence $\mu^*(p_1p_2 \cdots p_k) = (-1)^k$ is true for all k.

Next, let $n = p^2$. Then

$$\begin{aligned}\sum_{d/n} \mu^*(d) &= \mu^*(1) + \mu^*(p) + \mu^*(p^2)\\ &= 1 + (-1) + \mu^*(p^2) = 0.\end{aligned}$$

Hence $\mu^*(p^2) = 0$. If $n = p_1^2 p_2$, then

$$\sum_{d/n} \mu^*(d) = \mu^*(1) + \mu^*(p_1) + \mu^*(p_1^2) + \mu^*(p_2) + \mu^*(p_1p_2) + \mu^*(p_1^2p_2)$$

$$= 1 + (-1) + 0 + (-1) + (-1)^2 + \mu^*(p_1^2p_2)$$

by what has been proved above;

$= 0$ by definition of the μ^* function.

Hence

$$\mu^*(p_1^2p_2) = 0.$$

We shall again show by induction that $\mu^*(n) = 0$ if n has a squared factor. Let us assume that $\mu^*(n) = 0$ for all n such that $n = p_1^{\alpha_1}p_2^{\alpha_2} \dots p_r^{\alpha_r}$ where $\alpha_1 + \alpha_2 + \dots + \alpha_r < \lambda$ and at least one of the α's is greater than one. Note that here no restriction is imposed on r; but it can be seen that r can at most be equal to $\lambda - 1$. On this assumption we shall show that $\mu^*(n) = 0$ for all n such that

$$n = p_1^{\alpha_1}p_2^{\alpha_2} \dots p_r^{\alpha_r}$$

where

$$\alpha_1 + \alpha_2 + \dots + \alpha_r = \lambda + 1$$

(and hence when $\alpha_1 + \alpha_2 + \dots + \alpha_r < \lambda + 1$) and at least one of the α's is greater than one. Let $N = p_1^{\beta_1}p_2^{\beta_2} \dots p_k^{\beta_k}$ where $\beta_1 + \beta_2 + \dots + \beta_k = \lambda$ and at least one of the β's is greater than one. Let $N' = Np$. Now there are two possibilities.

Case (i). p is any one of the $p_1, p_2, \dots, p_k$; say p_i.

Then
$$N' = p_1^{\beta_1}p_2^{\beta_2} \dots p_i^{\beta_i+1} \dots p_k^{\beta_k}.$$

Now
$$\sum_{d/N'} \mu^*(d) = \mu^*(1) + \sum_{1}^{k} \mu^*(p_j) + \dots + \mu^*(p_1p_2 \cdots p_k) + \sum_{d'/N'} \mu^*(d') + \mu^*(N')$$

where d' is any divisor of N' having at least one squared factor and the total number of prime factors (not necessarily distinct) of d' is $< \lambda$ and for all such d', $\mu^*(d') = 0$ by induction hypothesis; and

$$\sum_{d/N'} \mu^*(d) = 0$$

by definition of μ^*. So we have

$$1 + (-1)^1\binom{k}{1} + \dots + (-1)^k + \mu^*(N') = 0$$

i.e.

$$(1-1)^k + \mu^*(N') = 0 \quad \text{or} \quad \mu^*(N') = 0.$$

Case (ii). p is different from all of $p_1, p_2, \dots p_k$. Then let $p = p_{k+1}$. Hence $N' = p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k} p_{k+1}$ and

$$\sum_{d/N'} \mu^*(d) = \mu^*(1) + \sum_{1}^{k+1} \mu^*(p_j) + \dots + \mu^*(p_1 p_2 \cdots p_{k+1})$$
$$+ \sum_{d'/N'} \mu^*(d') + \mu^*(N') = 0$$

where d' is governed by the same conditions as in the previous case. For each such d', $\mu^*(d') = 0$ by induction hypothesis. Hence

$$\sum_{d/N'} \mu^*(d) = (1-1)^{k+1} + \mu^*(N') = 0$$

i.e. $$\mu^*(N') = 0.$$

In each case N' has a total of $\lambda + 1$ prime factors and at least one of them occurs more than once. So the inductional hypothesis is true when λ is replaced by $\lambda + 1$. Since the hypothesis has been verified to be true for the case $\lambda = 3$, it is true for all values of λ; i.e. $\mu^*(n) = 0$ when n has a squared factor.

Hence the function μ^* satisfies (1) — (3) of the theorem and hence μ^* is identical with μ of the usual definition.

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2941. Lines associated with a triangle

For a triangle ABC denote by X_1, Y_1, Z_1 the points of contact with BC, CA, AB of the escribed circle opposite A, with corresponding notation for the other escribed circles. It is known that AX_1, BY_2, CZ_3 are concurrent and we denote the point of concurrence by L. One of our pupils, J. C. Longhurst, has proved that L is collinear with G (the centroid) and I (the incentre) and that $LG = 2GI$. He obtained this result by proving that AX_1 is parallel to the line joing I to the midpoint of BC.

Longhurst's result, which is new to us, has led us to a more general investigation. Through A, B, C draw lines parallel to the opposite sides to form the triangle $A''B''C''$. Then triangle $A''B''C''$ is inversely homothetic to triangle ABC with homothetic centre G and constant -2. Thus to prove Longhurst's result we have to show that L is the incentre of triangle $A''B''C''$.

Let x, y, z be real numbers with non-zero sum and let P be the C.M.P. (centre of mean position) of multiples x, y, z at A, B, C, so that x, y, z are proportional to the areal coordinates of P. Also let P'' be the C.M.P. of multiples x, y, z at A'', B'', C''. Then P'', G, P are collinear and $P''G = 2GP$.

We wish to associate P'' more directly with triangle ABC. To this end let P'' be the C.M.P. of multiples λ, μ, ν at A, B, C. Since A is the midpoint of $B''C''$ we may replace the multiple λ at A by

multiples $\frac{1}{2}\lambda$ at each of B'', C'', with like results for μ, ν. Thus

$$x = \tfrac{1}{2}(\mu + \nu), \quad y = \tfrac{1}{2}(\nu + \lambda), \quad z = \tfrac{1}{2}(\lambda + \mu),$$

and so

$$\lambda = y + z - x, \quad \mu = z + x - y, \quad \nu = x + y - z.$$

To deduce Longhurst's result take $x = a$, $y = b$, $z = c$. Then P coincides with I. Also

$$\lambda:\mu:\nu = s - a:s - b:s - c.$$

Since $$BX_1 = s - c, \quad X_1C = s - b,$$

and so on, it follows that P'' coincides with L. In view of relations such as

$$AB + BX_1 = X_1C + CA = s$$

Longhurst suggests calling L the *semicentre of triangle ABC*.

As a further example let P coincide with I_1, the excentre opposite A. Then

$$x:y:z = -a:b:c,$$

so that $$\lambda:\mu:\nu = -s:s - c:s - b.$$

It follows that P'' is the meet of AX, BY_3, CZ_2, where X is the point of contact of the incircle with BC.

To prove Euler's famous result let P coincide with O, the circumcentre. Then

$$x:y:z = \sin\ 2A:\sin\ 2B:\sin\ 2C,$$

leading to $$\lambda:\mu:\nu = \tan\ A:\tan\ B:\tan\ C.$$

Thus P'' coincides with H, the orthocentre.

As a final example let P be the symmedian point. Then

$$x:y:z = a^2:b^2:c^2$$

and $$\lambda:\mu:\nu = \cot\ A:\cot\ B:\cot\ C.$$

Hence if AH, AP'' meet BC in D, D'' then $BD = D''C$, with like results for the other sides. Thus P'' is the isotomic conjugate of H for triangle ABC.

We may obtain further results by considering the triangle $A'B'C'$ with vertices at the midpoints of BC, CA, AB. If P' is the C.M.P. of multiples x, y, z at A', B', C' then P', G, P are collinear and $GP = 2P'G$. To associate P' more directly with triangle ABC we have only to note that P' is the C.M.P. of multiples $y + z$, $z + x$, $x + y$ at A, B, C. We have so far failed to deduce from this any new result of special interest.

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CLASS ROOM NOTES

66. On note 2895

A very interesting lesson on generalization, formulae and the use of brackets can be developed from the starting-point of a few well-known Pythagorean number-triples. In each of the following series the first three sets of triples can be written on the board and the class encouraged to continue by analogy. A 3c Secondary Modern class did so successfully, and noticed several other interesting facts, but of course were not capable of producing the formulae.

The first series has successive odd numbers for the side a.

n	a	b	c	$(a+b+c)$	factors	p	q
1	3	4	5	12	3×4	2	1
2	5	12	13	30	5×6	3	2
3	7	24	25	56	7×8	4	3
4	9	40	41	90	9×10	5	4
5	11	60	61	132	11×12	6	5

The following formulae can be guessed, written down, and the various relationships between them proved:

$$a = 2n + 1 \qquad b = 4 \times (\text{series of triangular numbers } 1, 3, 6, 10\ldots) = 2n(n+1).$$

$$c = b + 1 = 2n(n+1) + 1; \; a + b + c = 2(n+1)(2n+1).$$

$$a^2 = b + c; \; b = n(a+1) = (n+1)(a-1).$$

$$p = n + 1, q = n; \; a = p^2 - q^2, b = 2pq, c = p^2 + q^2.$$

In the second series we take successive multiples of 4 for a.

n	a	b	c	$(a+b+c)$	factors	p	q
1	4	3	5	12	$2 \times 2 \times 3$	2	1
2	8	15	17	40	$2 \times 4 \times 5$	4	1
3	12	35	37	84	$2 \times 6 \times 7$	6	1
4	16	63	65	144	$2 \times 8 \times 9$	8	1
5	20	99	101	220	$2 \times 10 \times 11$	10	1

In this case

$$a = 4n, b = (2n-1)(2n+1) = 4n^2 - 1, c = b + 2 = 4n^2 + 1;$$
$$a + b + c = 2 \,.\, (2n)(2n+1); \qquad a^2 = 2(b + c); \qquad b = na - 1,$$
$$c = na + 1.$$

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67. On classnote 26

For a triangular (isosceles) lamina floating upright in a fluid, Mr. Srivastava uses a vector method to derive the equations

$$(f-e)\{2a\cos^2\alpha-(f+e)\}=0 \tag{1}$$

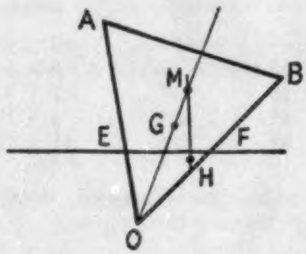
$$ef=a^2\sigma \tag{2}$$

where $e=OE$, $f=OF$, $a=OA=OB$, and σ is the S.G. of the lamina.

Equation (1) admits of two real, though not necessarily stable, solutions:

$$e+f=2a\cos^2\alpha \tag{3}$$

$$e=f \tag{4}$$



Equation (4) is intuitively evident, (3) is the equivalent of equation (1) in Note 26.

From (2) and (3), e, f are roots of

$$t^2-(2a\cos^2\alpha)t+a^2\sigma=0$$

The discriminant of this is

$$(\cos^4\alpha-\sigma)$$

so that if $\sigma=\cos^4\alpha$,

$$e=f=a\cos^2\alpha=a\sqrt{\sigma}$$

as expected from (2). However, if $\sigma>\cos^4\alpha$, the only real solution of (1) is $f=e$. To examine the stability of this case, consider the non-equilibrium situation shown in the figure. Then, by vectors

$$OM=OH+HM$$

and M is the Metacentre of hydrostatics. Then the vector HM is perpendicular to the vector EF, so

$$\begin{aligned} OM\,.\,EF=OM\,.\,(OF-OE)&=OH\,.\,EF+HM\,.\,EF\\ &=\tfrac{1}{3}(OE+OF)\,.\,(OF-OE)\\ m(f-e)\cos\alpha&=(f^2-e^2)/3\\ m&=(f+e)/3\cos\alpha \qquad f\neq e \end{aligned}$$

where $m=OM$.

Although we may not allow* $f = e$, we may take them as near as we like, so that, approximately

$$e \approx f = a\sqrt{\sigma} \qquad e + f = 2e = 2a\sqrt{\sigma}$$

Hence, writing

$$OG = g = \tfrac{2}{3}a \cos \alpha,$$

$$m = g\sqrt{\sigma}/\cos^2 \alpha$$

we see that stability requires $m > g$; hence the condition $e = f$ is stable if, and only if,

$$\sigma > \cos^4 \alpha$$

which is just the condition for failure of equation (3). Moreover, if $\sigma < \cos^4 \alpha$ $m < g$, so that the solution $e = f$ is then unstable and equation (3) gives the (stable) equilibrium position.

Note. Since $ef = a^2\sigma = \text{const}$, EF is tangent to an hyperbola asymptotic to OA, OB, and which is therefore the 'envelope' of the waterline. Deriving the above results from this fact is a healthy exercise in co-ordinate geometry.

University of Southampton D. R. BLACKMAN

THE MATHEMATICAL ASSOCIATION

The fundamental aim of the Mathematical Association is to promote good methods of Mathematical teaching. Intending members of the Association are requested to communicate with one of the Secretaries. The subscription to the Association is 21s. per annum and is due on January 1st. Each member receives a copy of the *Mathematical Gazette* and a copy of each new Report as it is issued.

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* Clearly, when $f = e$, the point M is indeterminate, which is the geometrical significance of division by 0 in this case.

CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

DEAR SIR,

In his article "Two problems on Impulsive Motion" (page 95 of the Mathematical Gazette, May 1960) Mr. O'Keeffe writes "the process of taking moments about a moving joint B (O'?) is valid only in cases where the second term of the equation

$$[\mathbf{H}(O')]_{t_0}^{t_1} + \int_{t_0}^{t_1} \mathbf{V}(O') \wedge \mathbf{L}\, dt = \mathbf{X}(O') \tag{2}$$

vanishes; the most important of which is ..." (In this equation $\mathbf{H}(O')$ is the angular momentum of the system about O', $\mathbf{V}(O')$ is the velocity of O', $\mathbf{L}$ is the linear momentum of the system and $\mathbf{X}(O')$ is the sum of the moments about O' of the applied impulses.) In fact the second term *always* vanishes, in the limit as $t_1 \to t_0$, since it is clear from the equation

$$[\mathbf{L}]_{t_0}^{t_1} = \mathbf{J},$$

where $\mathbf{J}$ is the sum of the applied impulses, that although $\mathbf{L}$ has jump-discontinuity at t_0, in the limit, nevertheless it is bounded in the interval (t_0, t_1). It follows, therefore, that the second term of equation (2) always vanishes and there is no restriction, when considering impulsive motion, on the validity of taking moments about a moving point.

Viewed in another way, impulsive motion is concerned with instantaneous change in the particle-velocities, and the equations of impulsive motion state the equivalence of two sets of localised vectors, the first set being the vectors which represent the change in momentum of the system, and the second set being the applied impulses. The motion, or otherwise, of a point about which moments are taken is clearly irrelevant.

I agree with Mr. O'Keeffe that many solutions claiming to use Bertrand's Theorem use Kelvin's Theorem, in fact. The equations expressing Kelvin's Theorem are, of course, precisely Lagrange's equations of impulsive motion for the coordinates corresponding to which there is no generalised component of impulse. Bertrand's Theorem states that if a system is subjected to given impulses, the kinetic energy generated is greater than it would have been if the system had been subjected to the same impulses and also workless constraints. So in order to use Bertrand's Theorem to solve a problem it is necessary to apply to the system variable constraints, depending, say, on parameters x_i; these constraints must be such that they are capable, by variation of the x_i, of allowing all possible motions of the unconstrained system. The problem of the system subject to these variable constraints and the applied impulses must then be solved, and the resulting kinetic energy T evaluated as a function of the x_i and the applied impulses. Maximisation of T with respect to the x_i will then yield a solution of the unconstrained problem.

As an example, consider the simple problem of a uniform rod AB, of mass m and length $2a$, at rest on a smooth horizontal table. If the rod receives a horizontal impulse J, perpendicular to AB, at A, then it is easily shown that the instantaneous centre of the resulting motion is between A and B at the point C where $AC = 4a/3$, and that the kinetic energy generated is $2J^2/m$. In order to use (or illustrate) Bertrand's theorem we apply a variable constraint by smoothly pivoting, to the table, the point P of the rod at distance x from A. The resulting kinetic energy T is easily shown to be given by

$$T = 3J^2x^2/2m\,(4a^2 - 6ax + 3x^2).$$

Differentiation with respect to x gives

$$\frac{dT}{dx} = 3J^2\,ax(4a - 3x)/m\,(4a^2 - 6ax + 3x^2)^2,$$

showing that T has, of course, a minimum at $x = 0$ and a maximum value, of $2J^2/m$, when $x = 4a/3$, that is when P coincides with C, and the resulting motion is the same as in the unconstrained case.

The conclusion seems to be that Bertrand's Theorem is not of much assistance in the exact solution of problems, though it may be of use in finding a lower bound for the kinetic energy.

Yours etc., S. T. Cook

To the Editor of the *Mathematical Gazette*

SUBTRACTION AND DIVISION

Dear Sir,

In a discussion on Subtraction, I see that I am being quoted as an authority for some modern method which teachers of infants have found useful.

Let us be frank about the duties of a member of a committee. Is he to be obstructive about every detail outside his own experience? On a matter of sacred principle or deep conviction let him dig in his heels in passionate protest; but if we all do this about every detail of which we know or care but little, what is left but a mosquito-like swarm of minority-reports on trivialities?

Like most Victorians, I subtract by the outmoded method of the 19th century; but if A or B prefers something better suited to this enlightened age, let him have it: it is out of place for me to object. This is surely a case for easy tolerance.

But if you want something for me to gnash a tooth about, take those mouldy little figures that look like indices and aren't, baffling enough even when neatly printed on page 180† of the current issue, and utterly chaotic when smudged about by a heavy-fisted boy with a fat pen.

† $\dfrac{5^12\,7^36}{4}$

In good Queen Victoria's reign we had heads on the tops of our necks, and we used them for remembering "remainder 3". But if the modern neck is not so garnished, why not put down three fingers of the left hand on the desk and lift them off again when done with? It is not an exhausting procedure.

Then what about a remainder 6? Well, haven't you a back and a front to your hand? When you come to remainder 11, if you haven't grown a head by then it is simpler to use Long Division than to take your boots off and put toes on the desk.

After all, I have seen a boy use Long Division for dividing by 1, and get every figure of the answer right too—(but unfortunately in the reverse order.)

Yours etc., W. Hope-Jones

Shamley Green, Guildford.

To the Editor of the *Mathematical Gazette*

Dear Sir:

This letter is an announcement of a new magazine which will appear bimonthly starting February 1961, and which I think might be of interest to readers of the *Mathematical Gazette*. Its title will be *Recreational Mathematics Magazine* and it will be devoted to the lighter side of mathematics. It will include such sundry items as paper-folding, interesting number phenomena, constructions, word games, mechanical puzzles, chessboard problems, treatment of various lighter mathematical topics, brainteasers—in short, anything that can be included in the rather extensive fields of recreational mathematics and puzzles. Of course, *Recreational Mathematics Magazine* is not going to be a mere collection of half-page or one-paragraph puzzles, but will include articles about and discussions of the above mentioned topics and more.

The magazine will sell for 5s ($.70) per issue but the subscription rate is 25s ($3.50) for each year. Cheques should be made out to *Recreational Mathematics Magazine*. Any person getting a new subscriber will receive a free issue and every five new subscribers will earn him a free year's subscription. Anyone getting new subscribers should include his name and address along with those of the new subscribers. Payment will be made for published material. Articles from about 3000 to 4000 words are needed and payment will run from $20.00 to $40.00 (£7 3s to £14 6s) per 1000 words. Puzzles and problems will receive from $5.00 to $10.00 (36s to £3 12s.)

The editor is a member of the National Council of Teachers of Mathematics and of the American Chemical Society. He is, at present, a research chemist with the Atomic Energy Division of Phillips Petroleum Company in Idaho Falls, Idaho, working in the field of radioactive waste disposal. He is a former teacher of mathematics and chemistry, an alumnus of Western Reserve University in Cleveland, Ohio, and an incurable puzzle-fiend.

Yours etc., Joseph S. Madachy

To the Editor of the *Mathematical Gazette*

DEAR SIR,

I wish to bring to your notice a statement, in the Presidential address to the Mathematical Association, April 1958, by W. J. Langford [*Math. Gazette* October 1958], which needs re-examination. Speaking on the place of Mathematics in Secondary School Curriculum he said, "Only rarely is a deliberate distinction made between girls and boys; in Denmark's real-klasse Mathematics is compulsory for the boys and optional for girls, *while in India the girls are required to study only Arithmetic*",

Through correspondence with my friends in various parts of India I have gathered the following information. There are certain states (in India) where no distinction is made between girls and boys in the matter of mathematical curriculum. In some States (in India) Mathematics is compulsory for the boys and optional for girls; and in those States girls who take Mathematics have to Study the same curriculum as the boys. I have not been able to locate any State in India where girls are required to study only Arithmetic.

Yours etc., S. PARAMESWARAN

Kerala University, Trivandrum, India.

DEAR SIR,

In reply to Professor Parameswaran's query about my statement concerning Mathematical education in India, I can do no more than quote my source of information.

In 1956 I attended a Conference of the International Bureau of Education in Geneva and all delegates were given a printed document containing the replies from Ministries to certain questions related to Mathematical education in secondary schools. In the Indian report occurs the following:

"Secondary education generally comprises a three-year "Middle School" stage (the 6th to the 8th year of schooling, pupils of 11 to 14 years old) and a two-year or three-year "High School" stage (the 9th to the 10th or 11th year of schooling, pupils of 14 to 16 or 17 years old).

Mathematics is taught throughout both these stages. It is designated arithmetic, algebra and geometry, but in some states also includes elementary mechanics and trigonometry in the upper classes. It is compulsory for boys in all years; for girls, arithmetic is compulsory, but algebra, geometry, etc., are optional".

Yours etc., W. J. LANGFORD

WANTED

"WANTED DESPERATELY DOWN UNDER. To buy, borrow or hire The Distribution of Prime Numbers by A. E. Ingham (Cambridge Tract No. 30) Please write Air Mail to M. G. Greening, 5 Alameda St., Parkdale, S.11., Victoria, Australia. ALL postage refunded and payment made in advance."

REVIEWS

An Analytical Calculus. IV. By E. A. MAXWELL. Pp. ix, 288. 22s. 6d. 1957. (Cambridge University Press).

This is the final volume of the author's series on the Calculus. The first two volumes cover the standard theory of differentiation and integration of functions of a single variable up to first year University level, and Volume III extends this theory to functions of several variables. This final volume is devoted to the study of differential equations and those parts of analysis concerned with functions defined by infinite series and integrals.

The book is divided into three sections, headed respectively Ordinary Differential Equations, The Definition of Functions by Infinite Series and Integrals, and Laplace's Equation and Related Equations.

Section 1 comprises four chapters. The first deals with the standard types of equations of the form $y' = f(x, y)$, the second gives some general properties of linear differential equations, and the third deals with the solution of linear differential equations with constant coefficients by means of the operator D. The fourth chapter gives a most pleasing account of the solution of linear differential equations by means of integrals, a method which is too often totally neglected in books of this level.

The treatment in this Section is clear and systematic. In the paragraph dealing with the determination of the particular integral of the equation $L(D + b)z = f(x)$ when $f(x)$ is a polynomial (pp. 42–8), the reviewer would have preferred the author to give a constructive method of solution (e.g. Math. Gazette, 42(1958), 47) rather than simply to state without proof a general rule giving the form of the solution, but this is probably a matter of personal preference. It would perhaps have been desirable also in the second chapter to stress more strongly the linearity property

$$L(Au + Bv) = AL(u) + BL(v)$$

of the differential operator in a linear differential equation; at present this property is used implicitly, but is never stated explicitly.

Section 2 gives what is (as the author remarks in his introduction) virtually a freshman's course in analysis. The subjects covered are convergence and uniform convergence of infinite series, the theory of power series, and the properties of functions defined by integrals with finite or infinite limits; the starting-point is the Principle of Monotone Sequences, which is taken for granted. There are also here chapters on the solution of differential equations in series and on Fourier series. This Section provides a good introduction to rigorous analysis, at just the right pace for a first introduction. The chapter on the solution of differential equations in series gives, too, immediate point to the work on series.

There are one or two mild slips here which should not be permitted in a section dealing with rigorous analysis. Thus the proof of the result that the series $\Sigma a_n x^n$ and $\Sigma n a_n x^{n-1}$ have the same radius of convergence,

given on p. 131, shows only that the radius of convergence of the first series does not exceed that of the second. Again, in dealing with the case $l = 1$ in Cauchy's and D'Alembert's tests for convergence (p. 85), the author does not distinguish between the failure of a proof and the failure of a theorem. There is also one odd gap, in that the divergence of a series is not defined.

Section 3 contains an account of the transformations of Laplace's equation into different systems of coordinates, of the various differential equations which are obtained when we seek to solve Laplace's equation by separation of variables (e.g. Legendre's and Bessel's equations), and of the theory of spherical harmonics. These topics are scattered throughout most textbooks on advanced calculus, and the Science student will find this connected account particularly valuable.

The whole volume provides an excellent introduction to the problems which are encountered in the solution of differential equations. It does to a very considerable degree achieve the author's aim of "bridging the gap between the works used in schools and more advanced studies with their emphasis on rigour", and can be recommended as wholeheartedly as its predecessors.

T. M. FLETT

Mathématiques Modernes) $\cap$ (Enseignement Elémentaire. By LUCIENNE FELIX. Pp. viii, 136. 12 N.Fr. 1960. (Albert Blanchard, Paris).

Those who have read Mlle Félix's earlier book, now available in English under the title "The Modern Aspect of Mathematics" (Basic Books, N.Y.), will be familiar with her lucid style and practical outlook, and will naturally expect to find the same qualities in this new work. They will not be disappointed, for this is a delightful exposition, perfectly described by its title—the intersection (in the set-theoretical sense) of modern Mathematics with elementary teaching. Mlle Félix is obviously at home in the domain of both sets. She writes with knowledge of modern mathematical ideas and enlivens her treatment with apt illustrations from the class-room.

The book is divided into three parts. The first and longest deals with general mathematical structures—sets, relations, binary operations, functions, measures and quantifiers, and topology. The basic elements of modern abstract ideas are here passed rapidly and clearly in review, and are applied throughout to the fundamentals of mathematics teaching, with abundance of examples from the classroom and from everyday life. The second part applies this theory to number and space, and contains much of direct value to the teacher; for example there is a brief discussion of the knowledge and appreciation of Pythagoras' Theorem appropriate to children of varying ages, from 10 to 17. The last section contains one short chapter devoted to "détails pédagogiques", in which good and bad habits of teachers of elementary mathematics are pilloried. Much of this is peculiar to the French situation and language, but there are lessons for all of us.

In her introduction, the author says "Dare one say (croirait-on) that certain loose statements in arithmetic which are tolerated in the primary school persist and lead to mistakes in all later secondary courses and even at the University?". One may well say so, and that is the reason for this book. The author advises, not that the mathematical structures she discusses shall be explicitly taught to children, but that they shall be in the background of the teacher's knowledge, and that every opportunity should be taken to train the child to think logically and with understanding in a way that will not have to be unlearnt later on. "The first chapters of higher mathematics teach, in abstract form, precisely those things which the kindergarten mistress has to bring to the notice of her pupils in order to teach them to think." For this reason such a book as this should be required reading for everyone introducing children to "number".

H. Martyn Cundy

A First Geometry. By R. L. Bolt. Without answers, pp. 96, 3s. 6d. With answers, pp. 112, 4s. 0d. Teacher's Book 1959 (Dent), 4s. 0d.

This book is essentially a class-book of exercises for the pupil, mainly of the drawing and measurement, or calculation types, simple geometrical argument being gradually introduced.

After an early section on solids, use of instruments and elementary ideas of angle, some varied examples of scale drawing and parallel line facts are given.

Later in the book the three exercises on loci and one on envelopes are particularly good, and contain interesting realistic examples, though it is perhaps a pity that on p. 72 the author felt it necessary to have two insects crawling about on a triangle in order to find its circumcentre.

Further work on solids is also included and a chapter on "Operations" is new and stimulating, leading, as it does, to some of the ideas of modern algebra.

It is doubtful whether many teachers will find, as the author claims, that "the book contains sufficient material to occupy an average form for two years", but it would be good for the first year, and could profitably be retained as a supplementary book for later in the course.

A set of calculations based on Pythagoras' Theorem avoids the usual over-emphasis on the 3:4:5 triangle by setting out the values of the twenty-six square roots needed, at the head of the exercise.

H. Bromby

Common Entrance Arithmetic and Algebra. By D. G. Munir. Part I, pp. 136, 5s. 6d. Part II, 4s. 6d. Parts I & II complete without answers, pp. 235, 9s. 6d. Parts I & II complete with answers, pp. 269, 12s. 6d. 1960. (Methuen).

The title of this book declares its aim. Arithmetic and Algebra are treated as one subject and well integrated in the sets of examples of

which the book consists. No attempt to suggest methods is made, beyond that implied in the wording of the questions, and no model solutions are given. Teaching is wisely left to the teacher.

The examples are numerous and varied, covering all the relevant branches of elementary arithmetic, and as far as quadratic equations with irrational roots in algebra, including a short section on graphs.

The paper on which the book is printed is of high quality and the lay-out is pleasant and clear.

It is distressing, however, to see in a book published in 1960 so many many elaborate calculations involving unrealistic compound quantities. The farthing, which is already practically obsolete, and which is to be withdrawn from circulation in December of this year occurs frequently in the money sums, and while it is just possible to imagine a period of time, such as the length of life of a satellite, which might be measured in weeks, days, hours, minutes and seconds, it is difficult to see the usefulness of a quantity such as 123 miles 3 furlongs 4 chains 11 yards 1 foot 6 inches or 71 tons 12 cwts. 0 qrs. 166 lbs. 12 ozs. both of which are to be found on p. 29; and these are by no means isolated instances.

If the examinations [Common Entrance and Eleven Plus] for whose syllabuses this book is designed to cater, really require the candidates to add, subtract, multiply and divide such quantities, surely reform is overdue.

H. BROMBY

Cartesian Geometry of the Plane. By E. M. HARTLEY. Pp. xi, 324. 20s. 0d. 1960. (Cambridge University Press).

Plane coordinate geometry is a field which has been somewhat overworked by writers of textbooks. In spite of this, we may welcome the present book, for several reasons. Firstly, it has been written with great care and clarity. Secondly, the author always has in mind the difficulties which beginners in this subject experience, and has tried to meet them. Thirdly, the book is enlivened by frequent asides, sometimes historical references, sometimes references to other branches of mathematics.

I have only one very small criticism. On p. 248, the equation of a parabola

$$9x^2 - 24xy + 16y^2 + 20x - 10y + 24 = 0$$

is arranged in the form

$$\tfrac{1}{25}(3x - 4y + 2)^2 = -\tfrac{2}{25}(4x + 3y + 10),$$

and then the substitution

$$X = \frac{4x + 3y + 10}{5},$$

$$Y = \frac{3x - 4y + 2}{5}$$

is used to give $Y^2 = -\frac{4}{3}X$. Now this substitution does not correspond to the usual change of axes, because of the signs. It would have been better to change the sign of X or Y, preferably X, because the equation then becomes $Y^2 = \frac{4}{3}X$.

The book is primarily intended for those studying for advanced and scholarship level examinations, bit it would also be useful for those studying for a general degree.

E. J. F. PRIMROSE

Coordinate Geometry. By L. P. EISENHART. Pp. xi, 298. $1.65. 1960. (Dover Publications).

A Dover reprint of the original edition, which was reviewed in the Gazette, Vol. XXIV (1940), p. 151.

E. J. F. P.

Advanced Algebra, Part I. By E. A. MAXWELL. Pp. ix, 311 16s. 0d. 1960. (Cambridge University Press.)

Dr. Maxwell, having already written textbooks on geometry and calculus, now turns to algebra, and the result is a book which should prove very useful in sixth forms of grammar schools. The subject matter corresponds roughly to the syllabus for advanced level examinations, though it starts a little below this and goes considerably beyond it in places. Everything is explained very clearly, and there are many examples, some worked by the author.

It is difficult at this level to invent any new methods which are likely to be successful, but as far as I know Dr. Maxwell's method for expressing a rational function in partial fractions, when the denominator contains repeated factors, is new and an improvement on existing methods.

Several "warning examples" are given, to show the reader that what seems obvious may not be true. One of these I found rather puzzling. Dr. Maxwell gives examples of various sequences, one of which is 1,2,3,4,5,6, ..., and he says that its p^{th} term is p, a statement that the reader would readily accept. Now comes the "warning example". What is the fifth term of the sequence 1,1,1,1, ...? At the back of the book, we are told that it can be anything, for example 25 if the p^{th} term were $(p-1)(p-2)(p-3)(p-4)+1$. On this argument the p^{th} term of 1,2,3,4,5,6, ... could be $(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)+p$. One should surely adopt the same criterion throughout: in dealing with sequences, it is usual to assume that if the general term is not given then the simplest rule which fits the given terms should be adopted.

The first volume should be a great success, and we shall await the second volume with interest.

E. J. F. PRIMROSE

Introduction to Analytic Geometry and Linear Algebra. By ARNO JAEGER. Pp. xii, 305. $5.50. 1960. (Holt, Rinehart & Winston, New York).

This is an unusual book. The author's object is to develop linear algebra and geometry together, so that the student can see the motive

for introducing the various algebraic concepts, and so that the abstract algebraic ideas can be illustrated by more concrete geometrical examples. The main theme of the book is that of vectors, which are first introduced abstractly by defining vector spaces, and are then applied to geometry.

The author claims that a course based on this book has been found suitable for freshmen at a university. My impression is that a freshman would find the course difficult, but if he mastered it he would have at his disposal some of the most important ideas and techniques of modern mathematics.

E. J. F. PRIMROSE

Anschauliche Mathematik. II Teil: Algebra. By HERBERT NOACK. Pp. 164. 1960. (Ferdinand Hirt, Kiel.)

This is the second part of a series intended for teachers of mathematics in high schools (the first part, on geometry, was reviewed in the Gazette, Vol. XLIII, p. 318). The book does not attempt to range over the whole of algebra, but concentrates on finite groups.

After giving two simple examples, to give the reader something concrete to grasp, the author defines a group, and proves some elementary results, including those on permutation groups. He then describes in detail all the groups of orders 4, 6 and 8. One attractive feature of this is an ingenious way of displaying the relations of the subgroups to each other. The next section deals with the groups of transformations of polygons and polyhedra; here again there are many helpful diagrams. The book concludes with an examination of the symmetry of several types of artistic patterns.

It may not be very long before we start to teach some group theory in our sixth forms. Provided that it is clearly presented, with plenty of examples, it should be no more difficult than some topics in the traditional syllabus. Meanwhile, any teacher who needs something to interest sixth-formers when examinations are over would find ample material here.

E. J. F. PRIMROSE

Dynamics. By A. E. SHORT. Pp. 382. 30s. 0d. 1959. (University of London Press Ltd.)

This book has been written mainly for all levels of sixth-form work in schools, but it will certainly be found useful, as the author hopes, to many students in the early part of their degree course. The extent of the material and the use of vector methods help considerably towards achieving these two ends.

The first two parts of the book cover the kinematics of a moving point and the dynamics of a particle, including chapters on resisting media, central orbits and problems of variable mass; the final part deals with the plane kinematics of a lamina and the motion of a rigid body. it is good to see a definite attempt to treat plane kinematics in such a text, but I would make a plea here for the teaching, also, of the allied

topics of displacements, finite and infinitesimal, real and virtual, and degrees of freedom.

The rigid dynamics is based on D'Alembert's Principle and includes sections on initial motions and impulsive motion. I do not think that this latter section was the best place to insert the principles of conservation of linear and angular momentum, but what is most unfortunate is the failure to point out that when calculating the rate of change of angular momentum the axis fixed.

The material is, in general, thoughtfully presented and it must be a great help to the pupil meeting vectors for the first time that the vector proofs usually supplement those in component form; but the dangers of too long a weaning process should not be overlooked.

Examples, including worked ones, are plentiful and well related to the chapters. Answers are also provided. This book should be considered by sixth-form teachers when making a choice of textbooks, if only because of the breadth of topics presented in one book at this level.

R. Buckley

The Solution of Equations in Integers. By A. O. Gelfond. Pp. 72. fl. 3,75. 1960. (P. Noordhoff. Groningen.)

This is a translation from the Russian by L. F. Boron of a little book based on a Lecture given at the Mathematics Olympics in Moscow in 1951. The greater part of the book is taken up with a complete account of the solution of Pell's equation, written in very simple terms and containing a new proof of the completeness of the familiar solution. Other topics considered include the impossibility of solving such equations as $x^2 - 3y^2 = -1$, $x^4 + y^4 = z^2$ and $x^4 + 2y^4 = z^2$, and an outline of the proof of Thue's Theorem.

This little book should be in every school library.

R. L. Goodstein

A First Course in Modern Logic. By E. W. Schipper and E. Schuh. Pp. 398, 28s. 1960. (Routledge and Kegan Paul; London.)

This is an extremely good book, in design, execution and production. The aim of the authors has been to emphasize principles of reasoning as they are expressed in modern forms of logic. To this end a very large number of examples is included. Since the exposition proceeds step by step each principle can be grasped both in application to arguments and in its logical relation to other principles. No attempt is made to express the relations among principles formally. That is to say there is no systematic exposition of axiomatic calculi. While this makes the book admirably suited to Arts students, it is not a text for mathematicians. However if one's philosophical colleagues should ask for a book which one would recommend for their first year students in logic one could hardly do better than bring this book to their notice.

R. Harré

The Use of Reason. By E. R. Emmet. Pp. 236. 10s. 6d. 1960. (Longmans.)

This book is admirably conceived and, on the whole, well executed. The author's aim is to produce a readable account of the principles of right thinking for sixth formers and the intelligent layman. In his chapters on 'Words', 'Certainty and Probability', 'Induction', 'Errors and Deceptions in Reasoning', and 'Solving Problems' he is most successful. His account of these topics is sensible, clear, and built round excellent examples. However the chapters on formal logic are far from being so successful. It is not that Mr. Emmet makes positive mistakes but that the formal chapters are spoiled by injudicious choice of methods and a disturbing lack of clarity in essential distinctions. For instance he chooses to express arguments involving class-relations in terms of Euler circles, which are at best only pictures of propositions, and have none of the essentially algorithmic properties of Venn diagrams. Lack of clarity about technical distinctions is most evident in his treating conditional statements as arguments; rather than as the principles of arguments which have the antecedent as premise and the consequent as conclusion. This leads the reader (and I suspect at times the author) to be confused between the truth of propositions and the validity of arguments, since arguments are only valid if the conditional statements which are their principles are true. It is to be hoped that in future editions of this potentially most useful book these defects are put right.

R. Harré

Logic in Elementary Mathematics. By R. M. Exner and M. F. Rosskopf. Pp. 274. 1959. (McGraw-Hill.)

The authors describe one of their aims as developing logic for the sake of explaining mathematics, rather than as a system in its own right, and they have been very successful in showing the part which a training in logic can play in helping a student to understand subtle distinctions in mathematics, like that between "$\phi(n) = 0$ from some n onwards" and "$\phi(n) = 0$ for infinitely many n".

The system of natural inference on which the authors rely is a rather dangerous one, and certain additional safeguards will need to be introduced when the work is revised. For instance the rule *PGE* (page 166) allows one to pass from

$$x \neq y \vdash x \neq y$$

to

$$x \neq y \vdash \exists x(x \neq x)$$

which is obviously false. And the deduction theorem (for predicate logic) needs the qualification that in the deduction no substitution occurs in the free variables of the assumption formula to be discharged by the theorem. Although an analysis of the properties of a *single* group may obviously be accomplished within first order predicate logic, the attempt to study the isomorphism of groups within this limited logic is not successful; introducing a special function $\phi(x)$ for the mapping, and constants e, e^* for the units of the two groups G, G^*, fails

to provide the possibility of proving $\phi(e) = e^*$ and the supposed proof on page 195 appears to be invalid. For if we postulate group axioms with e as unit, then

$$xy = x \to y = e$$

is certainly provable (page 188) and hence

$$\phi(a)\phi(e) = \phi(a) \to \phi(e) = e$$

follows, and since ϕ is supposed to satisfy

$$\phi(xy) = \phi(x)\phi(y),$$

from which $\phi(a)\phi(e) = \phi(a)$ is derivable, we have

$$\phi(e) = e, \qquad \text{not} \qquad \phi(e) = e^*.$$

If we seek to separate the variables, using x^* etc., for variables over G^*, with duplicated axioms, and allow $\phi(x)$ to be substituted for x^* but not for x, then we can show that for instance $\phi(e) = e^*$, but now we cannot discuss automorphisms.

R. L. Goodstein

Mathematics, Education and Industry. The Liverpool Conference Pp. 156. 7s. 6d. 1960. (The Times.)

This is a report of the Conference of Teachers, Research Scientists and Industrialists held at the University of Liverpool in April 1959, under the Chairmanship of Professor L. Rosenhead. In addition to many valuable contributions from Mathematicians in Industry which form a valuable guide to teachers advising students on careers, there is an account of mathematics in the U.S.S.R. and a discussion of the present shortage of mathematics teachers in this country which points out the danger to the schools which lies in the great opportunities open today for a research career in mathematics, the lack of which formerly obliged most of the abler mathematicians of the country to find a career in school teaching, to the great advantage of mathematical education.

R. L. G

Reflections of a Mathematician. By L. J. Mordell. Pp. 50. 1959. (Canadian Mathematical Congress.)

This little book, the preface tells us, had its origin in an after-dinner speech to members of the Royal Society of Canada. It is a book which every mathematician will enjoy reading, for its sincerity, its modesty, and its charm. What are the elements of a beautiful proof? "Obviously we desire a minimum of calculation and that of the easiest kind. The fundamental idea of the proof must be exceedingly simple, and just the right one for the problem considered. The application must seem inevitable and completely unexpected ... in rare instances the problem is

one whose origin goes back for many years and which seems so unapproachable that in the past no results have been found". Mordell sums up his life's work in these words:— "I can consider myself fortunate in having solved some really difficult and important problems in the course of my work. But vanity is chastened since there are a number of problems I have tried to solve again and again over many years but still in vain".

R. L. GOODSTEIN

The Memoirs of Selig Brodetsky. Pp. 323. 21s. 1960. (Weidenfeld and Nicolson.)

A brief but comprehensive obituary notice of the late Professor Selig Brodetsky appeared in the pages of the Monthly Notices of the Royal Astronomical Society in 1955 (**115**, 126–127) over the name of N. B. Slater. Now, five years later, Brodetsky's autobiography has been published, and it throws light on the social background of one who was well-known and well-loved in many parts of the world.

The writing does not run smoothly. The Memoirs are really selections from the masses of notes and papers that Brodetsky wrote during the illness which marred the last few years of his life. There is a breathlessness about many of the paragraphs which suggests a man racing with time. The rough manuscript was, in fact, completed only a few months before his death.

One day, in the early 1890's, during one of the recurrent periods of Russian violence directed against its Jewish minorities, a remarkable man, Akiva Brodetsky, smuggled himself out of the Jewish Ghetto of Olviopol. He went to find a new land, England, and a place for himself, his wife and his children. His family followed a little later and among them was his son Selig, later to become perhaps as remarkable as his father, and much more renowned.

In London the Brodetsky family found refuge from mental and physical oppression but they also found slums as fetid as those from which they had come. The father, with restless and fertile mind, was largely self-taught—classical Hebrew learning, mathematics, elementary science, philosophy, were all explored and stored away in his mind; he was so busy with learning that he did not appear to have been able to find time to earn a living for his wife and their brood. The story of young Selig's life in the East End of London may make strange reading for those who have had no contact with it. The past is a strange world and is peopled by even stranger characters.

It was from this family, and from these surroundings, that Selig grew up in the mentally fruitful atmosphere of the early 20th Century. He gained scholarship after scholarship and, by dint of mental ability and personal character, made a path for himself to the University of Cambridge. There, as the climax of a brilliant undergraduate career, he was bracketed with another student, and became one of the last two people ever to be crowned with the much-coveted title "Senior Wrangler", just before that designation was swept away by a radical reform of the mathematical instruction given in the University of Cambridge.

From that time onwards Selig Brodetsky was a man of many worlds; University Lecturer, University Professor, mathematician, aerodynamicist, astronomer, relativist, President of the Association of University Teachers, Zionist, statesman, ardent fighter for the causes of peace and universal brotherhood, and finally President of the newly-established Hebrew University of Jerusalem. It was this transition, from the slums of Olviopol to the University of Jerusalem, that prompted Selig Brodetsky to choose the words "From Ghetto to Israel" as the title of his autobiography. But it is not the grand sweep of this change of circumstance that will be recalled by those who knew him. What they will remember is a man surprisingly unsophisticated for all his apparent worldliness, a man twinkling-eyed and cherubic; warm-hearted and a great friend; devoted alike to his family, and to Israel and Great Britain, the countries which were interwoven in his dreams of a brave new world.

L. ROSENHEAD

The Elements of Determinants, Matrices and Tensors for Engineers. By S. AUSTEN STIGANT. Pp. xi, 433. 60s. 1959. (Macdonald.)

Mr. Stigant is well-known as an enthusiast for the electrical applications of the tensor calculus, so it is no disparagement to say that this is essentially a user's book; there is little of the striving for elegance and generality characteristic of the pure mathematician. Determinants are built up in easy stages from second to third and to higher orders, their structure and elementary manipulation are studied in great detail, with a wealth of simple numerical illustrations and exercises for the student. Straight-forward methods can be lengthy, and so, for instance, we do not reach the product rule till p. 59. Here I was surprised to find the rule suddenly thrust on the reader without any warning or reason given; my own experience with technicians at this level suggests that they will willingly dispense with a proof provided that the basic origins of a result are clear, but that they resent the invitation to be asked to "consider" some formula for which they have not been prepared. The study of matrices is on the same lines as that of determinants, and the simpler manipulations, such as diagonalisation and inversion, are set out in full, with plenty of examples.

These two sections are introductory to the second half of the volume, where simple tensor notations and properties are described and applied to electric networks. The author remarks that "it is most difficult to explain lucidly" what a tensor is save in abstruse mathematical language not altogether congenial to the engineer, and he attempts to overcome this obstacle by immediate reference to and illustration from network theory, with an immense number of clear and simple diagrams. This close weaving implies, I think, that no one save an electrical engineer would learn much about tensors from this treatment; but those who have faced the tedium which even a simple network problem can inflict will be grateful for this opportunity of acquiring in a congenial fashion the increased power which a more severe discipline can supply.

T. A. A. B.

An Introduction to Differential Geometry. By T. J. Willmore. Pp. x + 316. 35s. 1959. (Clarendon Press, Oxford University Press.)

The subject of Differential Geometry has tended to be neglected in undergraduate courses in England whereas on the Continent, especially in Germany, it has occupied a very important place. The many developments arising from the theory of relativity and the study of curved spaces at the research level do not seem to have penetrated to any great extent into undergraduate courses. The absence of a suitable textbook may have been largely responsible for this, and the appearance of Dr. Willmore's book should do much to remove this handicap. For those who need to know something about the curvature and torsion of curves, geodesics on surfaces, mean and gaussian curvature of surfaces, with such associated notions as minimal surfaces and developable surfaces, the first three chapters of this book will be found to contain what they need. Even in these preliminary chapters the reader will find the language and methods of modern mathematics in evidence.

In Chapter IV the author follows the example of Blaschke (Differential-geometrie Vol. 1) in the inclusion of a chapter on the differential geometry of surfaces in the large. Since the appearance of Blaschke's book however, quite a lot has happened particularly since the appearance of the papers by Hopf and Rinow in the early 1930's in which a definition of the notion of completeness as applied to surfaces was given. There are several topics in this chapter which appear for the first time in book form. The proof of the equivalence of the various definitions of completeness given by the author is the one appearing in a paper by de Rham in 1952. Also included in this chapter is a proof due to Bieberbach (1926) of Hilbert's theorem to the effect that a complete analytic surface free from singularities with constant negative gaussian curvature, cannot exist in three dimensional Euclidean space.

But the most important departure in this chapter lies in the treatment of intrinsically defined surfaces which prepares the way for the second part of the book. In the latter part of this Chapter IV he defines in modern terms the notions of manifold, equivalence classes of co-ordinate systems, orientability, triangulation, Riemannian structure and so on.

Part two of the book starts in Chapter V with vector spaces and the tensor product of vector spaces. In this there is a radical departure from the usual introduction of tensors in terms of the law of transformation of their components. This would have the effect of placing differential geometry nearer to the centre of a student's training in mathematics, in that differential geometry comes as a natural development from courses both in algebra and in analysis. In this fifth chapter there is included an introduction to exterior algebra, which, though originating from Grassmann, was shown to be a powerful instrument in the theory of Pfaffian systems, Lie groups and differential geometry by Elie Cartan. This subject has been rather neglected in English text books so far.

The author's contribution to bringing the modern ideas of differential geometry within the reach of undergraduate students is perhaps more

in evidence in Chapter VI than in any other. The notions of a differentiable manifold and of a tangent vector are introduced in a way which would lead a serious student quite naturally to an appreciation of the works of Ehresmann and to fibre bundle theory. When he comes to the theory of connections and of covariant derivation, his treatment comes nearer to more classical treatises on the same subject. This applies also to his treatment of Riemannian geometry in Chapter VII. In this chapter, which is the longest, there are several topics which appear in text book form for the first time such as recurrent tensors, integrable distributions and Riemannian extensions, all leading naturally to the section on global Riemannian geometry which ends the chapter.

The last chapter returns again to surface theory, treated this time by the methods of tensor calculus and exterior differential forms. I cannot help feeling that the contents of this chapter should have come at an earlier stage in the book, although it is not so easy to decide where.

There are ample references for further reading and there are plenty of exercises at the end of individual chapters as well as at the end of the book. The author's strong pedagogical sense is very much in evidence throughout the book. He has succeeded in writing an admirable text book which one can wholeheartedly recommend to students with differential geometry as part of their course.

E. T. Davies

Trigonometric Series. By A. Zygmund. Second Edition. 2 volumes, 84s. each. 1959. (Cambridge University Press.)

A new and up to date edition of Zygmund's famous book, long awaited and hoped for, is now with us. Ever since its first and only edition 25 years ago (three times reprinted) the book has been the only really comprehensive account of the theory of trigonometrical series, and of Fourier series in particular. Much has been added since then to our knowledge of the subject, not the least by Zygmund and his several pupils. I think that the mathematical world at large will be grateful to him for having undertaken the enormous task of rewriting the book, so that it is once more comprehensive and up to date. The subject is rather old and, in a way, more or less closed. There are a few major and apparently very difficult problems left, like that of "convergence almost everywhere" and problems of "uniqueness". Little progress has been made with these during those last 25 years, and the new results have been obtained rather at the "fringe" of the theory. It is, in the main, the development of new and more powerful methods of proof (like the "convexity method" of Zygmund and Calderon based on the M. Riesz-Thorin theorem) and the ingenuity of new techniques at the "fringe" which seem to be significant for that period. As so often in the earlier history of the subject these may prove to be signposts to future developments in wider fields.

Zygmund's book is, of course, not a textbook for the beginner, but is meant as a treatise for the connoisseur who will consider the new edition as a mathematical event of the first order, and who will not

quarrel about taste or value with those who look down anyhow on classical analysis as a spent force.

The first edition (Warsaw 1935) was rather badly printed and marred by many misprints. The new edition by the Cambridge Press, in two volumes, is printed beautifully. Moreover, the material is now much better arranged than before, and although the book quite naturally is not always easy to study, it is now as readable as can be expected even in its most difficult parts. New material can be found throughout the book. Of the new chapters in vol. II those on trigonometric interpolation, interpolations of linear operations, additional complex methods, Fourier integrals, and multiple Fourier series should be specially mentioned.

W. W. ROGOSINSKI

Theorie und Anwendung der Direkten Methode von Ljapunov. By W. HAHN. Pp. viii, 142. 41s. 6d. 1959. (Springer, Berlin.)

This is volume 22 of Springer's new Ergebnisse series and deals concisely and very clearly with Liapounoff's method for investigating the stability of solutions of ordinary differential equations and with its development in the last half-century. Liapounoff's great memoir was first published in Russian in the Commentary of the Kharkov Mathematical Society in 1893; a French translation appeared in the Annals of the Faculty of Science of Toulouse in 1907 and was reprinted ten years ago in the Princeton Studies of the Annals of Mathematics. The volume under review begins with an introduction and explanation of the fundamental ideas. Next are given sufficient conditions for stability and instability at a point of equilibrium and these are applied to concrete problems. Further chapters are concerned with the converses of the main theorems and with a number of further developments. It is very useful to have so lucid and compact an account of the literature of this subject whose interest and importance are still steadily increasing.

E. M. WRIGHT

Introduction to Probability and Statistics. By B. W. LINDGREN and G. W. MCELRATH. Pp. 277. 44s. 1959. (The Macmillan Company, New York.)

Probability is initially defined here through simple combinatorial problems, in which all the possible outcomes are supposed to be equally likely. The ideas of conditional probability and independence are then introduced. An account of discrete, mixed and continuous probability distributions follows; important distributions are singled out and means, variances, and percentiles derived. Basic results concerning a sum of independent variables conclude the first half of the book.

The authors now turn their attention to statistics. They discuss random sampling, presentation of data and the computation of sample means and variances. The Neyman-Pearson theory of testing hypotheses is stated. Tests and confidence intervals are given for the location

and scale parameters of individual distributions, also methods for comparing the parameters of two distributions. A final chapter looks so briefly at sequential tests, regression, analysis of variance and decision theory, that it might as well have been omitted.

A knowledge of integral calculus is assumed, but seems to be necessary only in the section on continuous distributions. Whereas nearly all the probability theory is proved, the statistical methods are mainly stated without justification, and, as indicated above, they do not go very far. Besides the classical material based on the normal distribution, there appear such comparative strangers as the tests of Kolmogorov-Smirnov and Wilcoxon-Mann-Whitney, to the second of which, incidentally, should be added the names of Haldane and Smith. The necessary tables are included. There are many problems, which often reflect the engineering interests of the authors, and answers are appended. The book is soundly based, written fairly well and has plenty of examples. It could support an introductory course for mathematicians at sixth-form or first-year level, although the price prohibits general purchase.

R. L. PLACKETT

Elementary Decision Theory. By H. CHERNOFF and L. E. MOSES Pp. 364. 60s. 1959. (John Wiley.)

Decision theory was originated in 1950 by the late Abraham Wald and it has since exerted a marked effect on the trend of statistical research in the United States, where statistics is now widely regarded as the science of decision making in the face of uncertainty. A typical problem facing a person "you" can be described informally in the following terms. You take an observation (x) at random from an unknown member (θ) of specified class of probability distributions, and then choose one member (a) of a set of available actions. A pure strategy (s) is any method of associating a single action with each possible observation, while a mixed strategy consists in adopting different pure strategies with different probabilities. For every combination of θ and a, you have already decided on the numerical value $l(\theta, a)$ of the loss of utility which you would suffer by taking action a when the state of nature, as it is called, is θ. The probabilities of the various actions are known when a state of nature and a pure strategy are given, and you can therefore use the values of $l(\theta, a)$ to compute the expected loss L as a function of θ and s. When this function is evaluated, you can rule out some strategies as inadmissible, because, when compared with an admissible strategy, they give rise to larger expected losses, whatever θ. No further simplification is possible and you must now select an appropriate criterion in order to determine which of the pure or mixed strategies are optimium. For example, one criterion consists in selecting the minimax strategy, defined as minimizing the maximum over θ of the expected loss. Another supposes that the values of θ have a specified prior probability distribution, in which case the Bayes strategy, which minimizes the expected value of $L(\theta, s)$ over θ, is preferred.

About half of the book under review is concerned with decisions within the general framework outlined above; the remainder describes the basic concepts of statistical theory and explains how the approach to a statistical problem is modified by the introduction of a utility function giving numerical values to the prospects with which a person may be faced. The authors devote much attention to definite, if often frivolous, problems with only a few values each of θ, x and a. As a result of this, and the relegation of awkward details to an appendix, the level of mathematical technique in the main text is quite elementary—no calculus is required—although mathematical ideas are introduced as they become necessary. One rather startling consequence of this approach is that the convex hull of a set of points makes it appearance before the equation for a straight line in Cartesian co-ordinates. However, why not? The first seven chapters constitute a first course; they are written with great clarity and a wealth of illustration. In the last three chapters, the standard rises appreciably and the last chapter is mostly a rapid survey of classical ideas on estimation without much regard to decision theory, but here the authors recommend some previous knowledge of statistics. There remain six mathematical and statistical tables, seventeen appendices which derive mathematical results only stated in the text, a partial list of answers to exercises, and an index.

The difficulties of applying decision theory are undoubtedly very real and, indeed, it is seldom applied at all outside its country of origin. Apart from the question of which criterion is appropriate to discriminate between the strategies, many people find it hard to assign numerical values to the utilities except in problems where monetary considerations prevail, and the same trouble has been experienced with prior probabilities ever since they were introduced by Bayes two hundred years ago. Consider, for instance, the following prospects, which occur in an example occupying the whole of the principal chapter, and which are assumed to lead to utility losses of 3 and 5 respectively: (a) wearing a raincoat, boots, rain hat and umbrella on a sunny day (b) wearing a fair-weather outfit on a rainy day. While perhaps in the right order, such losses seem hard to interpret numerically. Of course, the authors are fully aware of these objections, and point out that, when considerable data are available, small fluctuations in the losses or prior probabilities are unimportant; and no doubt they are right in implying that little can be concluded from few data. Whatever are the practical limitations of decision theory, the fact remains that Chernoff and Moses have written an excellent book, which can be strongly recommended to anyone teaching statistics.

R. L. Plackett

High Speed Computing: Methods and Applications. By S. H. Hollingdale. Pp. 244. 25s. 1959. (English Universities Press.)

This book is an account of electronic computers, their design and applications, for the general reader with a background in physics or mathematics. Following an introductory chapter describing the logical outline of a digital computer, the author discusses the representation of numbers. Here the reader is introduced to the binary scale, which is

used in most computers, and to such topics as multiple precision arithmetic, and floating point numbers. Chapter (3) introduces the basic ideas of programming, such as flow diagrams, cycles of operations, conditional jump instructions; and these are illustrated with the aid of easily understood processes, such as the evaluation of a square root. In Chapter (4), the author refers to the history of the subject and describes Charles Babbage's Analytical Engine, and the earlier American machines such as the ASCC (the first fully automatic calculating machine to actually work), and the ENIAC (the first *electronic* computer, the ASCC being electro-mechanical. The first electronic computer using a *stored programme* was the pilot machine developed by Professor F. C. Williams and Dr. T. Kilburn at Manchester University in 1948. This was thus the first truly *universal* electronic computer, since the ENIAC involved a plugboard programme, and had only a very limited storage capacity for numbers.

In chapter (5) there is a fairly detailed description of the EDSAC, one of the first British machines to be developed. The description includes an account of the programming system developed for the EDSAC, which as the author rightly emphasises, was pioneering work in this field. The DEUCE computer is given similar treatment in Chapter 6, and provides an interesting comparison to the EDSAC, since the two machines are of a very different nature. Chapter (7) is an account of storage devices and deals with delay lines (both mercury and magnetostrictive), the cathode-ray tube store, ferrite cores and magnetic drums and tapes. The next chapter deals with the logical design of computing circuits and their basic elements i.e., 'and' and 'or' circuits, gates, half adders etc. The alternative modes of serial and parallel operation are explained, and circuits for addition and multiplication are described. Chapter (9) deals with the operation of a computing service, and here the author draws on his 10 years experience as head of the mathematical services division at R.A.E., Farnborough. In this chapter the reader is shown into the "computer room"; and sees how programmes are prepared for running on the machine, "debugged" and finally reach the production stage, when they are handled by the operating staff. There is a section on computer maintenance, and the economics of automatic computing. The author briefly discusses the development of simplified programming techniques which enable people with no previous experience, to put their problems on a computer in a matter of days or hours. While the author undoubtedly appreciates the importance of these techniques, it is a pity that more space was not devoted to them, especially in view of the fact that nowadays a large number of users get their first experience of computers in this way. The next three chapters deal with applications of automatic computers. These include engineering application, such as bridge design, and the analysis of aircraft structures (much of which is essentially linear algebra); scientific applications such as the analysis of x-ray patterns to determine molecular structure; and real time applications involving the control of industrial processes. In this last section the author describes the final target of automation, the automatic factory in which the electronic computer takes over the complete control

of production, referring only to the sales and cost department for data and the management for policy decisions. While this goal has not yet been reached, certain sub-goals have been achieved, for example the computer controlled machine-tool system, an account of which is included. Finally there is a chapter on machine translation of languages, a subject which is as yet in the development stage, and is awaiting the arrival of computers with sufficient speed and storage capacity.

There is a selected bibliography which gives the more important papers in the subject. The reviewer can strongly recommend this book as an introduction to the subject, indeed it may become a standard work for some years hence.

R. A. Brooker.

Notebooks of Srinivasa Ramanujan, Tata Institute of Fundamental Research, Bombay. 2 vols. 4to boxed. 1957. 100 Rs. net.

These two volumes contain facsimiles of Ramanujan's unpublished notebooks. Professor K. Chandrasekharan has been responsible for their compilation, production and distribution, and financial assistance was provided by the Sir Dorabji Tata Trust. They form a very fitting memorial to the genius of their author and should be an inspiration to mathematicians who are interested in the formal side of mathematics. They may possibly stimulate a revival of this somewhat unfashionable subject. For Ramanujan was, perhaps, the last great formalist and his notebooks are packed with formulae of every description on such subjects as magic squares, prime number theory, definite integrals, continued fractions, infinite series, elliptic modular functions, and complex multiplication.

The notebooks date, presumably, from his early years, before his discovery as a mathematician and his arrival in England. It is clear that Ramanujan rediscovered a large number of results, many of which are of great difficulty; these sometimes appear in very unfamiliar guises. Apart from these, there are numerous formulae which he discovered for the first time. For a few results proofs, or indications of proof, are given, a few others are wrong, others have since been proved by Ramanujan himself, Hardy, Watson or others, while some still lack proofs. On skimming through the pages one comes across numerous formulae of the type of which Hardy wrote "I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have the imagination to invent them. Finally, the writer must be completely honest, because great mathematicians are commoner than thieves or humbugs of such incredible skill".

As stated in the preface to Ramanujan's *Collected Papers* (Cambridge, 1927), it would be a very formidable task to work through the notebooks systematically, selecting particular passages and editing these with adequate comment. Nevertheless it is a pity that the notebooks have been published without editorial comment of any kind, and the following brief remarks may help the reader in his study of them.

Apart from the *Collected Papers*, the fullest account of Ramanujan's work is to be found in G. H. Hardy's *Ramanujan* (Cambridge, 1940), which contains references to all the relevant literature before 1940. In particular, the following two papers give information about his notebooks: G. H. Hardy, A chapter from Ramanujan's notebook, *Proc. Cambridge Phil. Soc.* 21(1923), 492–503 and G. N. Watson, Ramanujan's notebooks, *J. London Math. Soc.* 6(1931), 137–153. Hardy's paper gives an analytical summary of the contents of Chapter XII of Notebook I, with comments and references to the existing literature on hypergeometric functions. Watson's paper gives a general account of the copies of the notebooks in his possession with numerous quotations and examples. Professor Watson, in his papers on singular moduli and other subjects, has done more than any other mathematician to supply proofs of results stated by Ramanujan.

A study of the facsimile edition in conjunction with Watson's paper allows one to make the following identifications, but leaves some questions unanswered. It is clear that Notebook I (published in the first of the two volumes) is Watson's "first edition." This is the notebook which Ramanujan left in Hardy's possession when he returned to India in 1919; it was returned to India later on in exchange for a copy.

Volume II of the facsimile edition contains Notebooks II and III. Notebook II appears to correspond to part of Watson's "second edition". It is not clear whether Notebook III also forms part of the second edition —corresponding perhaps to some of the "loose papers"—or whether it represents new material.

Notebook II is a kind of enlarged edition of Notebook I incorporating much of the miscellaneous matter at the end of Notebook I into the appropriate chapters. The first 256 pages of Volume II consist of 21 chapters of Notebook II, each listing in a systematic manner about 20 formulae or collections of formulae. Pp. 257–300 contain formulae fairly systematically arranged, but not divided into chapters; as with the miscellaneous matter at the end of Notebook I, they are mostly concerned with elliptic modular equations and transformations. Pp. 303–356 contain a variety of formulae, mainly on modular transformations, and are paged afresh in the manuscript from 1 to about 50. Finally Notebook III occupies pp. 361–393 and seems to consist partly of loose sheets. On pp. 368–9 it contains an extensive table which appears to consist of all numbers below 12006 that are composed entirely of the factors 1, 2, 3, 5 and 7. The three quarterly Progress Reports which Ramanujan wrote while holding a research scholarship at Madras in 1913–1914 are not included.

It would be of considerable interest to know when the various notebooks were written, but perhaps this is now impossible to ascertain. We must be thankful that they have been preserved and that their contents have now been made available to every mathematician. The price of the set of two volumes is reasonable but too high for the pocket of the average individual; it is to be hoped that they will be purchased by as many libraries as possible.

R. A. Rankin

La Dynamique Relativiste et ses Applications. By H. ARZELIÈS. Pp. 451. 60 NF. 1958. (Gauthier-Villars. Paris.)

Professor Arzeliès's earlier books in this series have been reviewed in *Math. Gaz.* **41**, 304, 1959, **43**, 148, 1959. The present book is mainly a systematic compilation of solutions of the equations of motion in special relativity of a single particle moving under the action of various types of force. Many of the solutions come from numerical integrations and are exhibited, and compared with the non-relativistic results, in numerous diagrams. Most applications of such work are to motions of charged particles, neglecting "radiative reaction", and some of these are to be treated in further volumes on particle-accelerators and electron-optics. The present volume closes with chapters on elastic and inelastic collisions between pairs of particles. Extensive bibliographies, mostly introduced by historical notes, are once again a feature of the work and the appendices include supplementary bibliographies for the earlier volumes.

W. H. MCCREA

The Theory of Elementary Particles. By J. HAMILTON. Pp. 482. 75s. 1959. (Oxford University Press.)

The title of this book is a misnomer. The book does not attempt to give a coherent connected theory of elementary particles—for no such theory exists; it is rather an account of the "basic mathematical methods used in investigating elementary particles, with physical examples to illustrate each method".

Accepting the conventional definition of an elementary particle, there are broadly speaking, two mathematical disciplines which have led to insights in elementary particle physics. These are respectively group theory, and mathematical analysis. Group theory has been used in the study of symmetry properties and conservation laws. These aspects of elementary particle physics lately emphasised (for example in the spectacular discovery of right-left asymmetry in physical law) one may call "kinematical" for lack of a better nomenclature. Mathematical analysis on the other hand (and particularly the theory of partial differential equations) has been used in the past to work out the detailed dynamics of elementary particles, which more recent work has been concerned with the general mathematical structure of the scattering matrix elements using the physical principle of relativistic causality, and employing theory of functions of many complex variables.

The "kinematical" aspects of the subject, among which one may include the study of spin, and the classification schemes of elementary particles, depend on basic principles of quantum mechanics as also do the analytical considerations based on the use of the principle of causality. These aspects of the theory are likely to have a permanence which the "dynamical" aspects of the subject by their nature cannot aspire to. This is firstly because the precise laws of interactions of elementary particles have been imperfectly guessed at, and secondly because the mathematical complexities of the equations of motion

which have been postulated are such as to defy any but approximate solutions. The current fashion in elementary particle theory is to make a severe separation of these two aspects of the subject, i.e. between the aspects based on general principles like symmetry and causality and those based on consideration of a particular set of dynamical equations of motion.

The volume under review deals with both aspects of the subject, but in common with most books already published such a separation is rather imperfectly attempted. This comes possibly from the style of work, a leisurely rather pleasant style, all too absent in modern texts. The book is in a sense, a minor encyclopaedia covering almost all topics known in elementary particle theory, and thus has considerable overlap with the works of Jauch and Rohrlich, Bethe and Schweber, Umezawa and Boguliubov on renormalization theory and general field theory, though additionally it includes also treatment of angular momentum, isotopic spin, strangeness, charge conjugation, parity violation, time reversal, β-decay theory etc. The large coverage means that all topics are not treated deeply; it, however, makes the book useful as an extensive survey of the subject.

One of the best chapters concerns polarization of particles. The references are unusually complete and generous.

A. Salem

Geometria dei Sistemi Algebrici Sopra una Superficie e Sopra una Varietà Algebrica. By Francesco Severi. Vol. III. Sviluppio delle teorie degli integrali semplici e multipli sopra una superficie e varietà e delle teorie collegate. Pp. VIII + 463. L. 4800. 1959. (Rome.)

It was in 1942 that Severi produced his book on series and systems of equivalence on an algebraic variety and invariant series of equivalence on an algebraic surface. This work was subtitled "volume primo", but it was not until 1958 that the second volume appeared; and in this the wider concept of algebraic equivalence was treated and the way was prepared for a discussion of the simple and multiple integrals attached to an algebraic variety. The volume at present under review, which concludes the set, takes up the discussion of the simple and multiple integrals and passes on to a number of interesting and related topics.

The book is subdivided into four main chapters which are advertised as being concerned with the fundamental theorem on the simple integrals the multiple integrals, Hodge's theorem on the periods of the integrals, of the first species, and the irregularities of a variety and the differential forms of the first species, respectively. However, none of the main chapter headings adequately describes the content. For example, in the second chapter, the consideration of the r-ple integrals of the first species on an r-dimensional variety leads the author, as it lead Noether, to the canonical system of hypersurfaces. But then the author forsakes the multiple integrals and goes on to discuss the algebro-geometric derivation of the canonical systems obtained by B. Segre, Todd, and himself, besides arithmetic genera and the virtual characters of (virtual)

varieties: there is also a section devoted to the anticanonical systems (the negatives of the canonical systems) which may exist effectively on varieties of geometric genus zero. (Incidentally on p. 128, there is a statement attributed to Castlenuovo and Enriques which should surely read "the linear connection of an M_r equals *twice* the superficial irregularity of the variety".) In contrast with this chapter, where the author actually deserts the subject of his main heading, the opening chapter (which deals with such things as the number of independent simple integrals of the first and second species, periods of the integrals, transcendental equivalence criteria and the Picard varieties attached to a variety), and the remaining two chapters, have a more predictable content.

In addition to the four main chapters there are six appendices. The three most closely connected with the rest of the book are the first, in which Severi gives an account of his efforts towards obtaining a Riemann-Roch theorem for series of equivalence (of linear circulation zero) on an algebraic surface, the second on the general theory of correspondences between a pair of algebraic varieties, and the last, contributed by Professor E. Marchionna, which contains an algebro-geometric proof of the Riemann-Roch theorem for a variety and a discussion of recent topological-transcendental methods.

This is a lengthy book, but its scope is vast and it is therefore in the nature of a survey. However, it is a very valuable survey because although, as Severi remarks in his preface, 1959 is his eightieth year, he maintains his lucid style and somehow keeps the reader interested in the broad pattern he traces. There are numerous references to original memoirs in the text which will supply the finer detail and, whilst the underlying spirit certainly seems to be classical, Severi from time to time makes reference to the more modern developments.

A. J. KNIGHT

Mathematische Werke. By ERICK HECKE. Ed. B. SCHOENEBERG. Pp. 955. 1959. (Vandenhoek and Ruprecht.)

This is a photographic reprint of all the published work of Hecke (1887–1947), excepting only his book on the Theory of Algebraic Numbers. It is prefaced by an obituary speech by J. Nielsen and a short introduction by C. L. Siegel. With a few exceptions, all the papers are concerned with the theory of modular functions and their application to the theory of numbers. In many cases the work is buttressed by his expert knowledge and liberal use of θ-series. Some outstanding results are:—

(i) In his dissertation (1912) Hecke introduced modular functions of two variables in order to construct absolute class fields of real quadratic number fields with the aid of complex functions.

(ii) The proof of the functional equation of the Dedekind Zeta Function (1917).

(iii) The introduction of Grössen Charaktere and the study of their L-series (1918–1920).

(iv) The explicit law of quadratic reciprocity in totally real algebraic number fields (1919).

(v) His systematic study of modular forms connected with Dirichlet series which possess an Euler product and satisfy a functional equation (1935–44).

The list is impressive. In assessing Hecke's work one is struck not only by the depth of his results but also by their vitality and importance for present research work in arithmetic theory. His proofs may strike a mathematician of the younger generation as laborious verification by calculation. But it is the common fate of the great pioneers that their results will be presented in later times more simply in a more abstract language which would have been unintelligble to the original workers in the field, and could not have been invented unless the spade-work had been done in the hard way originally.

H. Heilbronn

Mathematics Dictionary. By Glenn James and R. C. James. Pp. 546. 112s. 6d. 1959. (D. Van Nostrand Co., Ltd., London.)

This work is a development of two earlier Dictionaries, published in 1942 and 1949. In addition to an extension of the branches of mathematics whose basic terms are included, this new edition has a multilingual index, in French, German, Russian and Spanish. There are also tables of logarithms (five figure readings on both logarithms and antilogarithms), of trigonometric functions and annuities, long lists of integrals, and symbols grouped under topics. A sample test of definitions found them helpful and reliable.

R. L. Goodstein

BRIEF MENTION

The Real Projective Plane. By H. S. M. Coxeter. Students Edition. Pp. 226. 18s. 6d. 1960. (Cambridge University Press.)

This paper-back students' edition is a reprint of the second edition of 1955 of this celebrated book.

An Introduction to the Theory of Numbers. By G. H. Hardy and E. M. Wright. Fourth Edition. Pp. 421. 42s. 1960. (Oxford University Press.)

The main changes in the fourth edition of this famous book have been to bring the Notes at the end of each chapter up to date, to simplify the proofs of Theorems 234, 352, 357 and to add a new Theorem 272.

Ingenieur-Mathematik. By R. Sauer. Vol. I. Pp. 304. DM 24. 1959. (Springer, Berlin-Wilmersdorf.)

The first volume of this superbly produced work discusses the number concept, function and limit, differentiation and integration, convergent series, numerical and graphical methods, vector algebra and analytical geometry, curves and surfaces.

Algebras and their Arithmetics. By L. E. Dickson. Pp. 241. \$1.35. 1960. (Dover, New York. Constable, London.)

The Applications of Elliptic Functions. By A. G. Greenhill. Pp. 357. $1.75. 1960. (Dover, New York. Constable, London.)

Differential Equations for Engineers. By P. Franklin. Pp. 299. $1.65, 1960. (Dover, New York. Constable, London.)
Algebras and their Arithmetics is a reprint of the 1923 edition and is an excellent introduction to abstract algebra.
Differential Equations for Engineers includes ordinary and partial differential equations, and Fourier Series. Greenhill's *Elliptic Function* was written in 1892 and is a great store house of solved problems and examples.

A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. By E. T. Whittaker. Pp. 456. 30s. 1960. (Cambridge University Press.)

A paper-back edition of one of the great text books of the first decade of the century.

The Theory of Groups. By A. G. Kurosh. Translated by K. A. Hirsch. Second English edition. Vol. 1. Pp. 272. $4.95. Vol. 2. Pp. 308. $4.95. 1960. (Chelsea, New York.)

The appearance of the Second English Edition within four years of the publication of the first edition marks the success which this work so deservedly enjoys.

Grundlagen der Analysis. By E. Landau. Pp. 173. $1.95. 1960. (Chelsea, New York.)

The third edition of this famous little book contains no change. It is amusing to notice that in the *preface for the Student* (this German language edition) the translator of the preface makes Landau say "I will ask of you only the ability to read English".

Etude des sommes d'exponentielles. By L. Schwartz. 2nd Ed. Pp. 151. 1959. (Actualités scientifiques et industrielles 959. Hermann Paris.)

To the two chapters of the first Edition (1943) has been added a paper on approximation by imaginary exponential sums previously published in the Annales de la Faculté des Sciences de Toulouse.

The Theory of Functions of Real Variables. By J. Pierpoint. Vols. I, II. Pp. 560, 645. $2.45 each volume. 1960. (Dover, New York)

Functions of a Complex Variable. By J. Pierpoint. Pp. 583. $2.45. 1960 (Dover, New York)

A Course in Mathematical Analysis. By E. Goursat. Vol. I. Pp. 548. $2.25. Vol. II. Part 1. Pp. 259. $1.65. Vol. II, Part 2, Pp. 300 $1.65. 1960. (Dover, New York).

Of these three welcome reprints, Goursat's Analysis is too well known to need any recommendation; the reprint has been made from E. R. Hedrick's (1904) translation of Vol 1, and E. R. Hedrick and

O. Dunkel's translation of Vol. 2. (1916, 1917). Pierpoint's *Functions of Real Variables* reprints the 1905 Edition of Vol. I and the 1912 Edition of Vol. II. Apart from the familiar material of such a course, Volume II, contains accounts of the theory of transfinite cardinals and ordinals, Baire classes, and the Jordan curve theorem. The complex variable volume is a reprint of the 1914 Edition, and is chiefly concerned with the theory of special functions.

A Source Book in Mathematics. By D. E. Smith. Vols. I, II. Pp. 701, $1.85, each volume. 1960. (Dover, New York)

A reprint of the original 1929 Edition. A fascinating collection of selections from the works of the great (and not so great) mathematicians which all who love mathematics cannot fail to enjoy.

Essai sur la Psychologie de l'Invention dans le Domaine Mathématique. By J. Hadamard. Pp. 134. 8 N.F. 1960 (Blanchard, Paris)

A translation of "An Essay on the Psychology of Invention in the Mathematical Field," revised and augmented by the Author.

Theoretical Hydrodynamics. By L. M. Milne-Thompson. 4th Ed. Pp. 660. 65s. 1960. (Macmillan, London)

This fourth edition contains several additions: formulae of Peomels for solving boundary value problems; sections on flow under gravity with a free surface, surface waves of constant form and some comparison theorems.

Middel-Algebra. By D. P. Wijdenes. 6th Ed. Parts I, II. Pp. 419, 375. f17 each part. 1960 (P. Noordhoff, Groningen-Holland)

The ground covered is roughly that of a first year General Degree course.

Lectures on Fourier Integrals. By S. Bocher. Annals of Mathematics Studies No. 42. Pp. 333. $5.00. 1959 (Princeton, New Jersey)

This is a translation by M. Tenenbaum and H. Pollard of Bochner's classical work and itssupplement on Monotonic Functions, Stieltjes Integrals and Harmonic Analysis. The contents include the theory of positive definite functions, the generalised Fourier integral, and Fourier transforms.

String figures and other monographs. By W. W. Rouse Ball *et al.* Pp. 72, 102. 175, 136. $3.95. 1959. (Chelsea Publishing Company)

This reprint brings together four small books. The first which gives its title to the book, arose out of a lecture which Rouse Ball gave at the Royal Institution in 1920. The second is J. Petersen's *Methods and Theories for the solution of problems of Geometrical Constructions.* The third is H. S. Carslaw's *The Elements of Non-Euclidean Plane Geometry and Trigonometry* and the last Florian Cajori's *A History of the Logarithmic Slide Rule and Allied Instruments* which attributes the discovery of the Slide Rule to William Oughtred, whose instrument was described by William Forster in 1632.

Combinatorial Analysis. Proceedings of Symposia in Applied Mathematics. Vol. X. Pp. 311. 1960. (American Mathematical Society)

This volume contains 20 papers on very diverse subjects. There is a paper on finite division algebras by A. A. Albert, a paper by D. H. Lehmer on teaching combinatorial tricks to computers and a report on some computational work on machines by Olga Taussky and John Todd.

Finite Differences for Actuarial Students. By H. Freeman. Pp. 228. 17s. 6d. 1960. (C.U.P.)

This is a revised and abbreviated version of the second volume of the author's *Mathematics for Actuarial Students*, with an enlarged section on Miscellaneous Examples.

Vorlesungen über Differential-und Integralrechnung. Vol. 1. Funktiaren einer Variablen. 2nd. Ed. By A. Ostrowski. Pp. 330. Fr. 35. 1960. (Birkhäuser, Basel).

The first edition of this well known introductory text was reviewed by T. A. A. Broadbent in *Gazette* XXXI, p. 60. One of the changes in the new Edition has been to take out the collection of examples and prepare them for publication with solutions in a separate volume. Another change has been to simplify the treatment of real numbers by postulating a *separation axiom* which affirms the existence of a real number between any two classes of real numbers L, R such that every member of L is less than every member of R.

Proceedings of the Fourth Canadian Mathematical Congress. Pp. 184. 48s. 1960. (Toronto U.P. and O.U.P.)

The Congress was held at Banff in 1957. Amongst the invited speakers was H. S. M. Coxeter on *Factor Groups of the Craid Group* and P. Hall on the *Algebra of Partitions*. Outstanding amongst many reports on mathematical education is that by H. Zassenhaus on the education of graduate students.

An Introduction to Stochastic Processes. By M. S. Bartlett. Pp. 312. 22s. 6d. 1960. (C.U.P.)

A paper backed reprint of the first edition reviewed in *Gazette* XL, page 135.

Variational Principles in Dynamics and Quantum Theory. By W. Yourgrau and S. Mandelstam. 2nd Ed. Pp. 180. 32s. 6d. 1960. (Pitman)

The second edition contains a new chapter on the Feynman and Schwinger principles in quantum mechanics and some comments by Schrödinger. The first edition was reviewed in *Gazette* XL. p. 80.

Dynamics. By H. Lamb. Pp. 351. 18s. 6d. 1960.

A paper backed reprint of the second edition of this well known text-book.

Irratraialzahlen. By O. Perron. 4th Ed. Pp. 204. DM28. 1960. (de Gruyter, Berlin)

This new Edition of a valued book contains no major change but a number of corrections and small additions.

Vierstellige Tafeln und Gegentafeln. By H. SCHUBERT and R. HAUSSNER. 3rd Ed. Pp. 157. DM. 3.60. 1960. (de Gruyter, Berlin)

The present edition of this excellent book of tables in two colours was prepared by J. Erleback.

Vectoren und Matrizen. By S. VALENTINER. 2nd Ed. Pp. 200. DM. 580. 1960. (de Gruyter, Berlin)

This edition is an enlarged version of the authors well known book on vector analysis.

Partielle Differentialgleichungen. By G. HOHEISEL. 4th Ed. Pp. 130. DM. 3.60. 1960. (de Gruyter, Berlin)

Amongst the topics in this well stocked little book are linear partial differential equations in two and more variables, canonical transformations, contact transformations, linear systems in one and more unknowns, systems in involution and boundary value problems.

Algebra. By B. L. VAN DER WAERDEN. Part I. Vth Ed. Pp. 292. DM. 22. 1960. (Springer, Berlin)

Both the second and third editions of this celebrated book have received detailed reviews in the *Gazette* (XXI, p. 299; XXXV, p. 203). The present edition is substantially the same as the third and fourth.

The Unity of the Universe. By D. W. SCIAMA. Pp. 186. 21s. 1959. (Faber & Faber)

A lucid and original discussion of current cosmological theories.

Mathematical Snapshots. By H. STEINHAUS. Pp. 328. 48s. 1961. (Oxford University Press)

This new edition of a remarkable book contains about 25 per cent more material than the 1950 edition.

Solutions Numériques des Equations Algébriques. Tome I. E. DURAND. Pp. 328. 65 N.F. 1960. (Masson et Cie)

This first volume by the Director of the Applied Mathematics Centre of the University of Toulouse deals with the explicit equation $F(x) = 0$, and with roots of polynomials, found by various iterative methods.

Solutions of Equations and Systems of Equations. By A. M. OSTROWSKI. Pp. 202. 54s. 6d. 1960. (Academic Press)

A discussion of the Newton-Raphson formula, including existence conditions for the convergence of the successive approximations (in both the real and the complex case) and estimates of the error.

Handbuch der Schulmathematik. Vol I. By G. WOLFF. Pp. 295. DM. 38 (Hermann Schroedel Verlag KG. Hannover)

The first volume of this handbook contains sections by K. Wigand on numbers (decimals, powers, logarithms), complex numbers (trigonometry) and statistics; by J. Ladhoff on arithmetical and geometrical progressions; R. Mönkemeyer on number theory and J. Breuer in elementary set theory.

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